2026 Midterm Examination
Honors Complex Analysis (English)

1. Let f be a holomorphic function in the open unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$. If $|f(z)| \equiv 1$ for all $z \in D$, show that f is a constant function.

2. Consider the polynomial $p(z) = z^7 - 5z^2 + 1$. Prove the following statements:

(1) $p(z)$ has no roots on the circles $|z| = 1$ and $|z| = 2$.

(2) $p(z)$ has exactly 2 roots in the open region $|z| < 1$.

(3) $p(z)$ has no roots outside the closed region $|z| \leq 2$ (i.e., in the region $|z| > 2$).

3. Calculate the following integral:

$$\int_{-\infty}^{+\infty} \frac{e^{-2\pi i x\xi}}{x^2 + a^2}\, dx, \quad \xi \in \mathbb{R}.$$

4. Find all entire functions $f(z)$ such that for all $z \in \mathbb{C}$, the following inequality holds:

$$|f(z)| \leq |z|^3 \left(1 + |\operatorname{Re} z|^2\right).$$

5. Consider the function $f(z)$ defined by the following power series:

$$f(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n^2 + n}.$$

(1) Give an explicit formula for $f(z)$ in the domain $\{z \in \mathbb{C} \mid |z| < 1\}$.

(2) Find a point on the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ which is not a singularity of f.

6. Prove that there does not exist a holomorphic function f on the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$ that can be extended continuously to the unit circle, such that on the arc $\Gamma = \{e^{i\theta} \mid 0 \leq \theta \leq \varphi\}$ (where $0 < \varphi < 2\pi$), it satisfies

$$f(z) = \frac{1}{z}.$$

Solutions to 2026 Midterm Examination
Honors Complex Analysis (English)

1. **Problem:** Let f be a holomorphic function in the open unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$. If $|f(z)| \equiv 1$ for all $z \in D$, show that f is a constant function.

   Solution:
   By hypothesis, $|f(z)| = 1$ everywhere in D. Since f is holomorphic on D, its modulus $|f(z)|$ attains its local maximum (which is 1) at any interior point in D. According to the Maximum Modulus Principle, if the modulus of a non-constant holomorphic function attains a maximum in the interior of a domain, the function must be constant. Therefore, $f(z)$ must be a constant function.

   (Alternative proof: Since $|f|^2 = u^2 + v^2 = 1$, taking partial derivatives yields $uu_x + vv_x = 0$ and $uu_y + vv_y = 0$. Applying the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$, we get $uu_x - vu_y = 0$ and $vu_x + uu_y = 0$. Since $u^2 + v^2 = 1 \neq 0$, the only solution is $u_x = u_y = v_x = v_y = 0$, implying f is constant.)

2. **Problem:** Consider the polynomial $p(z) = z^7 - 5z^2 + 1$.

   Solution:

   (1) *Prove p has no roots on $|z| = 1$ and $|z| = 2$:*
   For $|z| = 1$: $|z^7 + 1| \leq |z|^7 + 1 = 2$. However, $|-5z^2| = 5|z|^2 = 5$. Thus, $|p(z)| \geq |-5z^2| - |z^7 + 1| \geq 5 - 2 = 3 > 0$. Hence, $p(z) \neq 0$ on $|z| = 1$.
   For $|z| = 2$: $|z^7| = 128$. And $|-5z^2 + 1| \leq 5|z|^2 + 1 = 21$. Thus, $|p(z)| \geq |z^7| - |-5z^2 + 1| \geq 128 - 21 = 107 > 0$. Hence, $p(z) \neq 0$ on $|z| = 2$.

   (2) *Prove p has exactly 2 roots in $|z| < 1$:*
   Let $f(z) = -5z^2$ and $g(z) = z^7 + 1$. On the boundary $|z| = 1$, we have $|f(z)| = 5$ and $|g(z)| \leq 2$. Since $|f(z)| > |g(z)|$ strictly on $|z| = 1$, by Rouché's Theorem, $p(z) = f(z) + g(z)$ and $f(z)$ have the same number of zeros inside $|z| < 1$. Since $f(z) = -5z^2$ has exactly 2 zeros (counting multiplicity at $z = 0$) in $|z| < 1$, $p(z)$ also has exactly 2 roots in $|z| < 1$.

   (3) *Prove p has no roots outside $|z| \leq 2$:*
   Let $F(z) = z^7$ and $G(z) = -5z^2 + 1$. On $|z| = 2$, $|F(z)| = 128$ and $|G(z)| \leq 21$. Since $|F(z)| > |G(z)|$ on $|z| = 2$, Rouché's Theorem implies $p(z) = F(z) + G(z)$ has the same number of roots as $F(z)$ in $|z| < 2$. $F(z) = z^7$ has 7 roots in $|z| < 2$. Since $p(z)$ is a degree 7 polynomial, all of its 7 roots must lie in $|z| < 2$. Consequently, there are no roots in $|z| \geq 2$.

3. **Problem:** Calculate $\int_{-\infty}^{+\infty} \frac{e^{-2\pi i x\xi}}{x^2+a^2}\, dx, \quad \xi \in \mathbb{R}.$

Solution:
Without loss of generality, assume $a > 0$. Consider the complex function $f(z) = \frac{e^{-2\pi iz\xi}}{z^2+a^2}$, which has simple poles at $z = ai$ and $z = -ai$.

Case 1: $\xi < 0$. We close the contour in the upper half-plane. Let Γ_R be the semi-circle of radius R in the upper half-plane. For $z = x + iy$ with $y > 0$, $|e^{-2\pi iz\xi}| = e^{2\pi y\xi} \le 1$ since $\xi < 0, y > 0$. By Jordan's Lemma, the integral over Γ_R vanishes as $R \to \infty$. The enclosed pole is $z = ai$.

$$\text{Res}(f, ai) = \lim_{z\to ai}(z - ai)\frac{e^{-2\pi iz\xi}}{(z - ai)(z + ai)} = \frac{e^{2\pi a\xi}}{2ai}$$

Thus, $\int_{-\infty}^{\infty} f(x)dx = 2\pi i \,\text{Res}(f, ai) = \frac{\pi}{a}e^{2\pi a\xi} = \frac{\pi}{a}e^{-2\pi a|\xi|}$.

Case 2: $\xi > 0$. We close the contour in the lower half-plane (clockwise orientation). The enclosed pole is $z = -ai$.

$$\text{Res}(f, -ai) = \frac{e^{-2\pi a\xi}}{-2ai}$$

Thus, $\int_{-\infty}^{\infty} f(x)dx = -2\pi i \,\text{Res}(f, -ai) = \frac{\pi}{a}e^{-2\pi a\xi} = \frac{\pi}{a}e^{-2\pi a|\xi|}$.

Case 3: $\xi = 0$. The integral is elementary: $\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}dx = \frac{\pi}{a}$.

In conclusion, for any $a \ne 0$ and $\xi \in \mathbb{R}$, the integral is $\frac{\pi}{|a|}e^{-2\pi|a||\xi|}$.

4. **Problem:** Find all entire functions $f(z)$ s.t. $|f(z)| \le |z|^3(1 + |\text{Re}\, z|^2)$.

   Solution:
   Since $|\text{Re}\, z| \le |z|$, we have $|f(z)| \le |z|^3(1 + |z|^2) = |z|^3 + |z|^5$. By the Generalized Liouville's Theorem, if an entire function grows at most like a polynomial of degree 5, it must be a polynomial of degree at most 5. Let $f(z) = c_0+c_1z+c_2z^2+c_3z^3+c_4z^4+c_5z^5$.

   As $z \to 0$, the bound $|f(z)| \le O(|z|^3)$ implies $f(0) = f'(0) = f''(0) = 0$, hence $c_0 = c_1 = c_2 = 0$. So $f(z) = c_3z^3 + c_4z^4 + c_5z^5$.

   Now consider points on the imaginary axis, $z = iy$ where $y \in \mathbb{R}$. For these points, $\text{Re}\, z = 0$. The given inequality becomes:

   $$|f(iy)| \le |iy|^3(1 + 0) = |y|^3 \implies |-ic_3y^3 + c_4y^4 + ic_5y^5| \le |y|^3$$

   Dividing by $|y|^3$ for $y \ne 0$, we get $|-ic_3 + c_4y + ic_5y^2| \le 1$ for all $y \in \mathbb{R}$. Taking $y \to \infty$, the left side will blow up unless $c_4 = 0$ and $c_5 = 0$.

   Therefore, $f(z) = c_3z^3$. Substituting this back into the original condition:

   $$|c_3z^3| \le |z|^3(1 + |\text{Re}\, z|^2) \implies |c_3| \le 1 + |\text{Re}\, z|^2 \quad (\forall z \ne 0)$$

   Setting $\text{Re}\, z = 0$ yields $|c_3| \le 1$. Hence, every such entire function is of the form $f(z) = cz^3$, where $c \in \mathbb{C}$ and $|c| \le 1$.

5. **Problem:** $f(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n^2+n}$.

   Solution:

(1) *Explicit formula*:
Using partial fractions, $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. For $|z| < 1$:

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \frac{z^n}{n} - \sum_{n=1}^{\infty} \frac{z^n}{n+1} \\ &= -\log(1-z) - \frac{1}{z} \sum_{n=1}^{\infty} \frac{z^{n+1}}{n+1} \\ &= -\log(1-z) - \frac{1}{z}\left(-\log(1-z) - z\right) \\ &= 1 + \left(1 - \frac{1}{z}\right)\log(1-z) \end{aligned}$$

where log denotes the principal branch of the complex logarithm.

(2) *Find a point on $|z| = 1$ which is not singular*:
The principal branch $\log(1-z)$ is singular (branch point) at $z = 1$. For any other point z_0 on the unit circle $|z_0| = 1, z_0 \neq 1$, $1 - z_0$ does not lie on the branch cut $(-\infty, 0]$ of the principal logarithm. Thus, $f(z)$ can be analytically continued to a neighborhood of z_0. For example, $z = -1$ is not a singularity.

6. Problem: Prove there is no holomorphic function on $D = \{z \mid |z| < 1\}$ which extends continuously to the unit circle and $f(z) = 1/z$ on an arc Γ.

Solution (Using Analytic Continuation):
Assume for the sake of contradiction that such a function $f(z)$ exists. Define an auxiliary function:

$$h(z) = zf(z) - 1$$

Since $f(z)$ is holomorphic in D and continuous on $\overline{D}$, $h(z)$ is also holomorphic in D and continuous on $\overline{D}$.

On the boundary arc $\Gamma = \{e^{i\theta} \mid 0 \leq \theta \leq \varphi\}$, we are given that $f(z) = \frac{1}{z}$. This means for all $z \in \Gamma$, $h(z) = z \cdot \frac{1}{z} - 1 = 0$.

Notice that $h(z) = 0$ is purely real on Γ. By the **Schwarz Reflection Principle** (a method of analytic continuation across circular arcs), we can analytically continue h across Γ to the exterior of the disk by defining:

$$H(z) = \begin{cases} h(z), & |z| \leq 1 \text{ (near } \Gamma) \\ \overline{h(1/\bar{z})}, & |z| > 1 \text{ (near } \Gamma) \end{cases}$$

The extended function $H(z)$ is holomorphic in an open domain Ω containing $D \cup \Gamma^\circ \cup D^*$, where Γ° is the open arc and D^* is the exterior region reflected across Γ°.

Since $H(z) = 0$ for all $z \in \Gamma^\circ$, and Γ° contains an accumulation point inside the domain of holomorphy Ω, we invoke the **Identity Theorem** for analytic functions. The theorem forces $H(z) \equiv 0$ everywhere in Ω.

Consequently, $h(z) \equiv 0$ for all $z \in D$. This implies:

$$zf(z) - 1 = 0 \implies f(z) = \frac{1}{z} \quad \text{for all } z \in D \setminus \{0\}$$

However, the function $1/z$ has a simple pole at $z = 0$, which directly contradicts our initial assumption that $f(z)$ is holomorphic in the entire disk D (which includes the origin).

Therefore, no such holomorphic function f can exist.