2026 Spring Complex Analysis (H, En) Final

1. Recall the Fourier transform:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} \, \mathrm{d}x$$

(1) Calculate $\hat{f}_a(\xi)$, where $f_a(x) = \frac{1}{\cosh(\pi a x)}$.

(2) Prove that:

$$\sum_{n=-\infty}^{\infty} \frac{1}{\cosh\left(\pi a \left(n + \frac{1}{2}\right)\right)} = \frac{1}{a} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\cosh\left(\frac{\pi n}{a}\right)}$$

2. Let f be an entire function satisfying $f(\Omega) \cap \Omega = \varnothing$ for any open set Ω with diameter < 1. Prove that $f(z) = z + c$, where $c \in \mathbb{C}$, $|c| \geq 1$.

3. (1) Prove that whenever $\Re(s) > 1$

$$M(s) := \zeta(s)\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, \mathrm{d}x$$

(2) You are given that

$$\frac{1}{e^x - 1} = \frac{1}{x} - \frac{1}{2} + \frac{1}{12}x + h(x), \;\; h(x) = \mathcal{O}(x^3) \text{ as } x \to 0.$$

Prove that $M(s)$ can be analytically continued to $\Re(s) > -3$ via the following formula:

$$M(s) = \int_1^\infty \frac{x^{s-1}}{e^x - 1} \, \mathrm{d}x + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} + \int_0^1 x^{s-1} h(x) \, \mathrm{d}x$$

(3) Prove that: $\zeta(0) = -\frac{1}{2}$, $\zeta(-1) = -\frac{1}{12}$, $\zeta(-2) = 0$, and $\operatorname{Res}_{s=1} \zeta(s) = 1$.

4. Let f be an entire function satisfying:

(1) $\exists C > 0, \; 0 < B < \pi$ such that $|f(x+iy)| \leq Ce^{B|y|}$,

(2) $f(n) = 0, \; \forall n \in \mathbb{Z}$,

Prove that $f \equiv 0$.

5. (1) Consider $q(z) = z + i\sqrt{z^2 - 1}$ defined on $\Omega = \mathbb{C} \backslash ((-\infty, -1) \cup (1, \infty))$. Prove that q conformally maps Ω onto the upper half-plane $\mathbb{H}$.

(2) Find a conformal mapping $\varphi : \Omega \to \mathbb{D}$ such that $\varphi(0) = 0$ and $\varphi'(0) > 0$.

(3) Let $\Omega \subset \mathbb{C}$ be a domain. If a function $g(z, z_0)$ is harmonic in $\Omega \backslash \{z_0\}$, vanishes identically on the boundary $\partial\Omega$, and satisfies that $g(z, z_0) + \ln|z - z_0|$ can be harmonically extended near the pole z_0, then it is called the Green's function on Ω with

pole at z_0. In particular, when $z_0 = 0$, if $\varphi : \Omega \to \mathbb{D}$ is a conformal mapping satisfying $\varphi(0) = 0$, the Green's function can be directly written as $g(z, 0) = -\ln|\varphi(z)|$. Find the Green's function $g(z, 0)$ at $z = 0$ for the domain Ω.

(4) Investigating the boundary behavior, onto which arcs of $\partial\mathbb{D}$ does φ map $(1, \infty)$ and $(-\infty, -1)$ respectively?

6. Let $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, where $\tau \in \mathbb{C}$, $\Im(\tau) > 0$, and let P be a fundamental parallelogram. Suppose f is an elliptic function with no zeros or poles on ∂P. Let Q be the set of zeros and poles of f inside P. For $q \in Q$, denote $\mathrm{ord}_f(q)$ as the multiplicity of the zero or pole, which is positive for zeros and negative for poles.

(1) Recalling that you can use the residue theorem on $\int_{\partial P}$, prove that:

$$\sum_{q \in Q} \mathrm{ord}_f(q) = 0, \quad \sum_{q \in Q} q \, \mathrm{ord}_f(q) \in \Lambda$$

(2) Let a, b, c, d be four distinct points inside P. Suppose f has simple zeros at a, b and simple poles at c, d. Prove that $a + b - c - d \in \Lambda$.

(3) Conversely, given $a + b - c - d \in \Lambda$, construct an elliptic function f that has simple zeros at a, b and simple poles at c, d.