Combinatorics, Midterm Exam, 2026-2027 Fall

**** Choose as many questions as you can to solve. But you can get at most 100 points in total. Your answers could be in English or in Chinese.****

**** You can use all results proved in classes, but if you want to use some homework problems then you need to give a full proof. ****

1. (12)

(a) Give a combinatorial proof that for integers $0 \le m \le n$,

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m} 2^{n-m}.$$

(b) Prove that for every $n \ge 1$,

$$\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

2. (12)

(a) Use the Inclusion–Exclusion Principle to find the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 = 20, \qquad 0 \le x_i \le 8 \ (i = 1,2,3,4).$$

(b) Use the Inclusion–Exclusion Principle to prove that the number of surjections $f : [n] \to [3]$ is $3^n - 3\cdot 2^n + 3$ $(n \ge 1)$. Deduce a closed formula for the Stirling number of the second kind $S(n,3)$.

3. (16)

(a) Let a_n be the number of ways to tile a $1 \times n$ rectangle using 1×1 squares, each colored either red or blue, and 1×2 rectangles, all colored red. Put $a_0 = 1$. Compute the ordinary generating function of $\{a_n\}$, and derive a closed formula for a_n.

(b) Let T_n be the number of words of length n over the alphabet $\{a,b,c\}$ in which the number of a's is odd and the number of b's is even. Use exponential generating functions to find T_n.

4. (8) Let G be the graph on $\{1,2,3,4\}$ obtained from K_4 by deleting the edge $\{3,4\}$. Write down its Laplace matrix Q and use Kirchhoff's Matrix-Tree Theorem to compute the number of spanning trees of G.

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5. (16) Let $G = (V, E)$ be a simple graph with n vertices and e edges.

(a) Prove by double counting that $\sum_{v \in V} d(v)^2 = \sum_{uv \in E} \big(d(u) + d(v)\big)$.

(b) Let $t(G)$ be the number of triangles in G. Prove that

$$t(G) \;\geq\; \frac{e\,(4e - n^2)}{3n}.$$

[Hint: count pairs (edge, triangle containing it) and use (a).]

6. (16)

(a) Let M be an $n \times n$ 0-1 matrix in which every row and every column contains exactly $k \geq 1$ ones. Prove that M is the sum of k permutation matrices.

(b) Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be two partitions of the same set X with $|X| = nk$ into n blocks each, where every block has size exactly k. Prove that there exist distinct elements $x_1, \ldots, x_n \in X$ and a permutation σ of $[n]$ such that $x_i \in A_i \cap B_{\sigma(i)}$ for all i.

7. (15) Let $(X, \mathcal{B})$ be a symmetric (v, k, λ) design ($b = v$, $v > k \geq 2$, $\lambda \geq 1$) and let M be its $v \times v$ incidence matrix (rows indexed by points, columns by blocks).

(a) Prove that $r = k$ and $MM^T = (k - \lambda)I + \lambda J$, where J is the all-ones matrix.

(b) Prove that $k > \lambda$ and $\det(MM^T) = k^2(k - \lambda)^{v-1}$.

(c) Prove that if v is even then $k - \lambda$ is a perfect square. Deduce that there is no symmetric $(22, 7, 2)$ design (first check that these parameters satisfy $b = v$).

8. (15) Let $(X, \mathcal{B})$ be a (v, k, λ) design with b blocks and replication number r (i.e., each point appears in r blocks), and fix a block $B \in \mathcal{B}$. For $i \geq 0$ let n_i be the number of blocks $B' \neq B$ with $|B \cap B'| = i$. Prove

$$\sum_i n_i = b - 1, \qquad \sum_i i\, n_i = k(r - 1), \qquad \sum_i \binom{i}{2} n_i = \binom{k}{2}(\lambda - 1).$$

Then assume the design is symmetric ($b = v$) and use these identities to prove that any two distinct blocks meet in exactly λ points.

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