2026 年随机过程中期试卷

姓名 ________________ 学号 ________________ 学院 ________________

题号	一	二	三	四	五	六	总分
得分							

一、【10 分】

Let $\mathcal{F}$ be a σ-field. Let X be a bounded random variable. Prove

$$E[X^2 \mid \mathcal{F}] \geq \big(E[X \mid \mathcal{F}]\big)^2.$$

(It is not allowed to use Jensen's inequality.)

二、【15 分】

Let $\{X_n, Y_n\}_{n\geq 1}$ be a sequence of square integrable random variables. Suppose that $\{S_n = \sum_{k=1}^n X_k\}_{n\geq 1}$ and $\{V_n = \sum_{k=1}^n Y_k\}_{n\geq 1}$ are martingales w.r.t. a family of σ-fields $\mathcal{F}_n$, $n \geq 1$. Prove that

$$Z_n = S_n V_n - \sum_{k=1}^n X_k Y_k, \qquad n \geq 1$$

is a martingale w.r.t. the family of σ-fields $\mathcal{F}_n$.

三、【25 分】

Let $X_n, n \geq 1$ be independent random variables with $P(X_n = 1) = p$ and $P(X_n = -1) = q = 1 - p$, where $p \neq q$. Let $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$ be the σ-field generated by $X_1, X_2, \ldots, X_n$ and $\mathcal{F}_0 = \{\Omega, \varnothing\}$. Consider the random walk $S_n = \sum_{i=1}^n X_i$, $n \geq 1$, with $S_0 = 0$. Suppose $0 < p < 1$. It is known that $\{Z_n = (\frac{1-p}{p})^{S_n},\ n \geq 0\}$ is a martingale with respect to $\mathcal{F}_n$, $n \geq 0$.

For an integer x, define the stopping time $T_x = \min\{n;\ S_n = x\}$, the first time at which the random walk hits the position x. For integers $a < 0 < b$, let $T = T_a \wedge T_b = \min(T_a, T_b)$.

(i) Determine the value of $E[Z_T]$ and give your reason.

(ii) Use the result in (i) to show that

$$P(T_a < T_b) = \frac{\phi(b) - \phi(0)}{\phi(b) - \phi(a)},$$

where $\phi(x) = \left(\dfrac{1-p}{p}\right)^x$.

(iii) Consider an appropriate martingale to find $E[T]$.

四、【15 分】

Box A, B, each contains m balls. Among the total $2m$ balls, m of them are white and the other half of them are black. In each round, we randomly pick a ball respectively from box A and B, and then we place the ball taken from box A in box B and place the ball taken from box B in box A.

Let X_0 be the number of white balls in box A at the beginning and X_n the number of white balls in box A after the n-th round. Then $\{X_n, n \geq 0\}$ forms a Markov chain with the state space $S = \{0, 1, 2, \ldots, m\}$. Find the one step transition probabilities $p(i, j)$, $i, j \in S$, of the Markov chain $\{X_n, n \geq 0\}$.

五、【15 分】

Let $(X_n, n \geq 0)$ be a Markov chain with state space S. Let $x \in S$ be fixed. Let $\tau_0 = 0$,

$$\tau_n = \inf\{m > \tau_{n-1};\ X_m = x\}.$$

Define $N = \sum_{m=1}^{\infty} \mathbf{1}_{\{X_m = x\}}$.

(i) Prove

$$P_x(\tau_n < +\infty) = P_x(\tau_1 < +\infty)^n$$

for $n \geq 1$.

(ii) Prove that following statements are equivalent.

(a). x is recurrent.

(b). $P_x(N = +\infty) = 1$.

六、【20 分】

Let $X = (X_n, n \geq 0)$ be a Markov chain with state space S and the transition matrix $(p(x, y))$. For $a \in S$, define $\tau_a = \inf\{k \geq 0;\ X_k = a\}$. For a bounded function h on the space S, define the operator P by

$$Ph(x) = \sum_{y \in S} p(x, y) h(y), \qquad x \in S.$$

(i) Let $\{\pi(x)\}$ be a stationary distribution. Prove that

$$\int_S Ph(x)\, \pi(dx) = \int_S h(x)\, \pi(dx),$$

namely

$$\sum_{x \in S} h(x)\pi(x) = \sum_{x \in S} Ph(x)\pi(x).$$

(ii) We say that a bounded function h is harmonic at the point x if $h(x) = Ph(x)$. Define $h(y) = P_y(\tau_a < \infty)$ for $y \in S$. Prove that h is harmonic at any point $x \neq a$.

(iii) Suppose that the Markov chain X is irreducible, i.e., $\rho_{xy} > 0$ for any $x, y \in S$. Let h be a non-negative bounded harmonic function at every point in the space S. Prove that if $h(x) = 0$ for some point x, then $h(y) = 0$ for every point $y \in S$.