2026 年随机过程期末试卷

姓名________　　学号________　　学院________

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一、　【15 分】

Let N_t, $t \geq 0$ be a Poisson process of rate λ. N_t denotes the number of customers who arrive at a certain supermarket by time t. Suppose that the probability that a customer buys something from the supermarket is $\frac{2}{3}$. Let $\widetilde{N}_t$ denote the number of customers who buy something from the supermarket by time t. Determine the probability $P(\widetilde{N}_t = k)$.

二、　【30 分】

Let $\{B_t\}_{t\geq 0}$ be a Brownian motion started from 0.

(i) Prove

$$\lim_{t\to\infty} \frac{B_t}{t} = 0$$

almost surely.

(ii) Define the stopping time

$$T = \inf\{t \geq 0;\ B_t \leq 2t - 2\}.$$

Explain why $T < \infty$ almost surely.

(iii) For $\lambda > 0$, determine the Laplace transform

$$E[\exp(-\lambda T)].$$

三、　【15 分】

Let $\{X_t, t \geq 0\}$ be an adapted process with continuous paths on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Let T be a finite stopping time. Show that X_T is $\mathcal{F}_T$-measurable.

四、　【15 分】

Let $\{X_t\}_{t\in[0,\infty)}$ be an $(\mathcal{F}_t)$-nonnegative supermartingale with right continuous paths. Prove that as $t \to \infty$, X_t converges almost surely to a random variable X_∞, and moreover

$$E[X_\infty \mid \mathcal{F}_t] \leq X_t, \qquad t \geq 0.$$

五、 【25 分】

Let $B = (B_t)_{t\geq 0}$ be a Brownian motion and define $X_t = |B_t|$, $t \geq 0$. For $t > 0$, we set

$$p_t(z) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{z^2}{2t}\right), \qquad z \in \mathbb{R}.$$

For a bounded measurable function f on $\mathbb{R}_+$, define the operator

$$Q_t f(x) = \int_0^\infty \big(p_t(y-x) + p_t(y+x)\big) f(y)\, dy, \qquad x \geq 0,$$

and $Q_0 f = f$. Let

$$\mathcal{F}_t = \sigma(B_u, u \leq t).$$

Prove that $(X_t)_{t\geq 0}$ is a Markov process with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ with the transition semi-group $(Q_t)_{t\geq 0}$, namely

$$E[f(X_{t+s}) \mid \mathcal{F}_s] = Q_t f(X_s)$$

for any bounded function f on $\mathbb{R}_+$.

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