Riemannian Geometry (Spring, 2026)
Mid-term Exam

Name: No.: Department:

Notations:

- $R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$;
- $R(W,Z,X,Y) := \langle R(X,Y)Z, W\rangle$.

In a local chart (U,x), we have

$$R^i{}_{k\ell j} = \partial_\ell\Gamma^i_{jk} - \partial_j\Gamma^i_{\ell k} + \Gamma^p_{jk}\Gamma^i_{\ell p} - \Gamma^p_{\ell k}\Gamma^i_{jp},$$

where we use the notation $\partial_\ell\Gamma^i_{jk} := \frac{\partial}{\partial x^\ell}\Gamma^i_{jk}$. Moreover, we have $R_{ik\ell j} := g_{ip}R^p{}_{k\ell j}$.

1. (20 marks) Let M be a smooth manifold, and let g and $\tilde{g}$ be two Riemannian metrics on M such that $\tilde{g} \geq g$ (i.e., $\tilde{g}(v,v) \geq g(v,v)$ for all tangent vectors v). Answer the following questions and provide reasoning for your answers.

   (i) If (M,g) is complete, must $(M,\tilde{g})$ be complete?
   (ii) If $(M,\tilde{g})$ is complete, must (M,g) be complete?

2. (15 marks) Let (M,g) be a Riemannian manifold. Let $\gamma : [0,\infty) \to M$ be a normal geodesic with $p = \gamma(0)$.

   (i) Give the definition of a cut point of p along γ.
   (ii) Suppose $\gamma : [0,a] \to M$ is a shortest curve connecting $p := \gamma(0)$ and $q := \gamma(a)$, and there exists another shortest curve connecting p and q. Show that q is the cut point of p along γ.

3. (15 marks) Let (M,g) be the upper half-plane $\{(x,y) \in \mathbb{R}^2 : y > 0\}$, assigned with the Riemannian metric

$$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy).$$

   (i) Show that the Laplacian satisfies

$$\Delta = f(x,y)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

   for some smooth function $f : M \to \mathbb{R}$, and determine the expression of f.
   (ii) Show that the distance function d satisfies

$$d((x_0,y_0),(x_0,y_1)) = \log\frac{y_1}{y_0}$$

   for any given $x_0 \in \mathbb{R}$ and $0 < y_0 \leq y_1$.

4. (15 marks) Let (M,g) be a complete Riemannian manifold, and let $N \subset M$ be a compact submanifold of M without boundary. Let $p_0 \in M \setminus N$, and let

$$d(p_0,N) := \inf_{q\in N} d(p_0,q)$$

be the distance from p_0 to N. Show that

(i) there exists a point $q_0 \in N$ such that $d(p_0, q_0) = d(p_0, N)$.
(ii) A minimizing geodesic $\gamma : [a, b] \to M$ which joins p_0 to q_0 is orthogonal to N at q_0, that is, $g(\dot\gamma(b), V) = 0$, for any $V \in T_{q_0}N \subset T_{q_0}M$.

5. (35 marks) Let (M, g) be an n-dimensional Riemannian manifold, $p \in M$.

(i) Use the exponential map $\exp_p$ to define a normal coordinate $(x^1, x^2, \ldots, x^n)$ centered at p.

(ii) Show that in the above normal coordinate,

$$g_{ij}(p) = \delta_{ij},\ \Gamma^k_{ij}(p) = 0,\ \text{and}\ \partial_k g_{ij}(p) = 0,$$

for any $i, j, k \in \{1, 2, \ldots, n\}$.

(iii) Show that in the above normal coordinate,

$$\partial_\ell \Gamma^k_{ij}(p) + \partial_i \Gamma^k_{j\ell}(p) + \partial_j \Gamma^k_{\ell i}(p) = 0,$$

for any $i, j, k, \ell \in \{1, 2, \ldots, n\}$.

(iv) Show that in the above normal coordinate,

$$\partial_j \Gamma^i_{k\ell}(p) = -\frac{1}{3}\left(R^i{}_{k\ell j}(p) + R^i{}_{\ell kj}(p)\right),$$

for any $i, j, k, \ell \in \{1, 2, \ldots, n\}$.

(v) Show that in the above normal coordinate,

$$\partial_\ell \partial_k g_{ij}(p) = -\frac{1}{3}\left(R_{ikj\ell} + R_{i\ell jk}\right),$$

for any $i, j, k, \ell \in \{1, 2, \ldots, n\}$.

(vi) Show that in the above normal coordinate, the following Taylor expansion holds

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikj\ell} x^k x^\ell + O(|x|^3),$$

for any $i, j \in \{1, 2, \ldots, n\}$.

(vii) In the above normal coordinate, for a small $r > 0$, consider the curve $C_r : [0, 2\pi] \to M$ such that

$$C_r(t) := (x^1(t), x^2(t), \ldots, x^n(t)),$$

with $x^1(t) = r\cos t$, $x^2(t) = r \sin t$, $x^k(t) = 0$ for $k \in \{3, \ldots, n\}$. Let L_r be the length of the curve C_r. Use (vi) to show that

$$\lim_{r \to 0} \frac{2\pi r - L_r}{r^3} = \frac{\pi}{3} K(\Pi_p),$$

where Π_p is the section of T_pM spanned by $\frac{\partial}{\partial x^1}(p)$ and $\frac{\partial}{\partial x^2}(p)$.