2025 ～2026 学年《数学分析B2》期中考试解答

一、(每小题6分, 共36分) 计算下列各题.

1. 求函数$f(x,y,z)=\left(\dfrac{x}{y}\right)^{\frac{1}{z}}$ 在$(1,2,1)$ 处的梯度及沿方向$\vec{e}=(1,-2,2)$ 的方向导数.

   解: 计算在点$M(1,2,1)$ 处的偏导数：

   $$\frac{\partial f}{\partial x}|_M=\frac{d}{dx}(\frac{x}{2})|_{x=1}=\frac{1}{2},\quad \frac{\partial f}{\partial y}|_M=\frac{d}{dy}(\frac{1}{y})|_{y=2}=-\frac{1}{4},\quad \frac{\partial f}{\partial z}|_M=\frac{d}{dz}(\frac{1}{2})^{\frac{1}{z}}|_{z=1}=\frac{\ln 2}{2}.$$

   方向向量$\vec{e}=(1,-2,2)$，单位化：$\vec{e^\circ}=\left(\frac{1}{3},-\frac{2}{3},\frac{2}{3}\right)$.

   方向导数：$\dfrac{\partial f}{\partial \vec{e}}=\nabla f|_M\cdot\vec{e^\circ}=\dfrac{1}{3}+\dfrac{\ln 2}{3}$.

2. 设$z=f(x,y)=\displaystyle\int_1^{xy^2}\frac{\cos t}{t}\mathrm{d}t$, 求微分$\mathrm{d}z$.

   解: $dz=\dfrac{\cos(xy^2)}{x}\,dx+\dfrac{2\cos(xy^2)}{y}\,dy$.

3. 求方程$f'_y(x,y)=\sin x+2y$ 满足$f(x,x^2)=1$ 的解$f(x,y)$.

   解: 对y 积分得

   $$f(x,y)=\int(\sin x+2y)\,dy=y\sin x+y^2+C(x).$$

   代入条件$f(x,x^2)=1$: $x^2\sin x+(x^2)^2+C(x)=1\Longrightarrow C(x)=1-x^2\sin x-x^4$

   所以$f(x,y)=y\sin x+y^2+1-x^2\sin x-x^4$.

4. 设函数$z=f(x-y,g(xy))$, 其中f 有二阶连续偏导数, g 有二阶连续导数, 求$\dfrac{\partial^2 z}{\partial x\partial y}$.

   解:

   $$\frac{\partial z}{\partial x}=f'_1\cdot 1+f'_2\cdot yg'(xy)=f'_1+yg'(xy)f'_2.$$

   $$\frac{\partial^2 z}{\partial x\partial y}=-f''_{11}+(x-y)g'(xy)f''_{12}+xy[g'(xy)]^2f''_{22}+f'_2\big[g'(xy)+xyg''(xy)\big]$$

5. 计算$\displaystyle\int_0^{2\pi}x\mathrm{d}x\int_x^{2\pi}\frac{\sin^2 y}{y^2}\mathrm{d}y$.

   解: 积分区域为$0\le x\le 2\pi$, $x\le y\le 2\pi$，交换积分次序得：

   $$I=\int_0^{2\pi}\frac{\sin^2 y}{y^2}\left(\int_0^y x\,dx\right)dy\cdots(3分)$$

   $$=\int_0^{2\pi}\frac{\sin^2 y}{y^2}\cdot\frac{y^2}{2}\,dy=\frac{1}{2}\int_0^{2\pi}\sin^2 y\,dy=2\int_0^{\frac{\pi}{2}}\sin^2 y\,dy=\frac{\pi}{2}$$

6. 计算$\displaystyle\iint_D xy\mathrm{d}x\mathrm{d}y$, 其中$D$是由$y=x^2$和$y=2x$所围成的区域.

   解: $I=\displaystyle\int_0^2 x\,dx\int_{x^2}^{2x}y\,dy=\int_0^2\frac{1}{2}x(4x^2-x^4)\,dx=(\frac{1}{2}x^4-\frac{1}{12}x^6)|_0^2=\frac{8}{3}$.

   第一步3分, 后面各1分

二、(本题10 分) 设函数$f(x,y)$ 有二阶连续偏导数, 满足$f_x'^2 f_{yy}'' - 2f_x' f_y' f_{xy}'' + f_y'^2 f_{xx}'' = 1$ 且$f_y' \neq 0$, 又$y = y(x,z)$ 是由方程$z = f(x,y)$ 所确定的函数, 求$\dfrac{\partial^2 y}{\partial x^2}$.

解: 由隐函数求导，对x 求偏导（z 固定）:

$$0 = f_x' + f_y' \cdot \frac{\partial y}{\partial x} \quad \Rightarrow \quad \frac{\partial y}{\partial x} = -\frac{f_x'}{f_y'}.$$

再对x求导得

$$f_{xx}'' + f_{xy}'' \frac{\partial y}{\partial x} + (f_{yx}'' + f_{yy}'' \frac{\partial y}{\partial x}) \frac{\partial y}{\partial x} + f_y' \frac{\partial^2 y}{\partial x^2} = 0$$

$$\frac{\partial^2 y}{\partial x^2} = -\frac{f_{xx}''}{f'_y} + \frac{f'_x f_{xy}''}{f_y'^2} + \frac{f'_x f_{xy}''}{f_y'^2} - \frac{f_x'^2 f_{yy}''}{f_y'^3} = -\frac{f_y'^2 f_{xx}''}{f_y'^3} + \frac{2f_x' f_y' f_{xy}}{f_y'^3} - \frac{f_x'^2 f_{yy}''}{f_y'^3}.$$

由已知条件得 $\dfrac{\partial^2 y}{\partial x^2} = -\dfrac{1}{f_y'^3}$. ······ (10分)

三、(本题10 分) 求过点$(1,1,1)$且与直线$l_1: x = \frac{y}{2} = \frac{z}{3}$, $l_2: \begin{cases} x - 2y - 5 = 0 \\ 4y - z + 11 = 0 \end{cases}$ 都相交的直线L, 及L绕X轴旋转一周所成的曲面方程.

解: l_1 过原点, 方向$v_1 = (1,2,3)$, l_2 过点$(1,-2,3)$, 方向$v_2 = (2,1,4)$.

过$(1,1,1)$ 且包含l_1 的平面Π_1: 法向$\vec{n}_1 = (1,1,1) \times (1,2,3) = (1,-2,1)$,

方程为 $(x-1) - 2(y-1) + (z-1) = 0$

过$(1,1,1)$ 且包含l_2 的平面Π_2: 法向$\vec{n}_2 = (0,3,-2) \times (2,1,4) = (14,-4,-6)$,

方程为 $7(x-1) - 2(y-1) - 3(z-1) = 0$

所求直线L的一般方程是 $\begin{cases} x - 2y + z = 0 \\ 7x - 2y - 3z = 2 \end{cases}$

消去x得 $y = \dfrac{5x-1}{4}$, $z = \dfrac{3x-1}{2}$

所求旋转面方程是

$$y^2 + z^2 = \left(\frac{5x-1}{4}\right)^2 + \left(\frac{3x-1}{2}\right)^2 = \frac{61x^2 - 34x + 5}{16}.$$

四、 (本题共12 分) 讨论$f(x,y)$在点$(0,0)$处的连续性、任意方向的方向导数的存在性及可微性, 其中

$$f(x,y) = \begin{cases} y \arctan \dfrac{1}{\sqrt{x^2+y^2}}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0. \end{cases}$$

解:(1)当$(x,y) \neq (0,0)$，记$r = \sqrt{x^2+y^2}$，则

$$|f(x,y)| = |y| \cdot \left| \arctan \frac{1}{r} \right| \leq r \cdot \frac{\pi}{2} \to 0 \quad (r \to 0^+),$$

故 $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$，因此$f$ 在$(0,0)$ 连续。

(2) 设方向$\vec{e} = (\cos\theta, \sin\theta)$，则

$$\lim_{t\to 0} \frac{f(t\cos\theta, t\sin\theta) - 0}{t} = \lim_{t\to 0} \sin\theta \arctan \frac{1}{|t|} = \frac{\pi}{2} \sin\theta \Rightarrow \frac{\partial f}{\partial e} = \frac{\pi}{2} \sin\theta$$

(3)

$$\theta=0\text{时},\ f'_x(0,0)=0,\quad \theta=\frac{\pi}{2}\text{时},\ f'_y(0,0)=\frac{\pi}{2}.)$$

令$\rho=\sqrt{x^2+y^2}$

$$\lim_{\rho\to 0}\frac{f(x,y)-0\cdot x-\frac{\pi}{2}\cdot y}{\rho}=\lim_{\rho\to 0}\frac{\rho\sin\theta\arctan\frac{1}{\rho}-\frac{\pi}{2}\rho\sin\theta}{\rho}=0$$

所以$f(x,y)-f(0,0)=f'_x(0,0)x+f'_y(0,0)y+o(\rho),\qquad(\rho\to 0)$,
$f(x,y)$在$(0,0)$可微.

五、(本题15 分) 求函数$f(x,y,z)=\ln x+\ln y+3\ln z$ 在条件$x^2+y^2+z^2=5r^2$ $(r>0)$, $x>0$, $y>0$, $z>0$ 下的最大值, 并证明对$\forall\, a,b,c>0$ 有$abc^3\leqslant 27\left(\frac{a+b+c}{5}\right)^5$.

解:原问题等价于求$u=xyz^3$在条件$x^2+y^2+z^2=5r^2, x,y,z\geqslant 0$下的最大值. 因为函数$u=xyz^3$在有界闭集$D=\{x^2+y^2+z^2=5r^2,\ x,y,z\geqslant 0\}$上一定有最值, 且$u\geq 0$. u在边界上取最小值零, 故最大值一定在$x,y,z>0$时取得.
作辅助函数$F(x,y,z)=xyz^3+\lambda(x^2+y^2+z^2-5r^2)$, 则有驻点方程组

$$\begin{cases}F'_x=yz^3+2\lambda x=0, & (1)\\ F'_y=xz^3+2\lambda y=0, & (2)\\ F'_z=3xyz^2+2\lambda z=0, & (3)\\ x^2+y^2+z^2-5r^2=0, & (4)\end{cases}$$

将(1), (2), (3)式分别乘x, y, z, 然后相加, 并利用(4)得$xyz^3=-2\lambda r^2$, 将其代入(1), (2), (3)式, 解得$x=y=r,\quad z=\sqrt{3}r$.

于是$u_{max}=3\sqrt{3}r^5$, 则函数f的最大值为$\ln(3\sqrt{3}r^5)=5\ln r+\frac{3}{2}\ln 3$.

由上面的结论$xyz^3\leqslant 3\sqrt{3}r^5$ 两边平方得$x^2y^2z^6\leqslant 27r^{10}$. 令$x^2=a,\ y^2=b,\ z^2=c$, 则$r^2=\frac{a+b+c}{5}$. 从而有$x^2y^2z^6=abc^3\leqslant 3^3r^{10}=27\left(\frac{a+b+c}{5}\right)^5$.

六、(本题共10 分) 设函数$f(u,v)$ 具有连续的偏导数, 且对实数$t>0$ 满足$f(tu,tv)=t^2f(u,v)$, $f(1,2)=0, f'_u(1,2)=3$, 求极限

$$\lim_{x\to 0}\frac{f(x-\sin x+1,\sqrt{1+x^3}+1)}{x^3}.$$

解:两边对t求导得:

$$f(tu,tv)=t^2f(u,v)\Longrightarrow uf'_u(tu,tv)+vf'_v(tu,tv)=2tf(u,v).$$

由于$f(u,v)$有连续偏导数，可知$t=1$时上式成立，代入点$(u,v)=(1,2)$，已知$f(1,2)=0$, $f'_u(1,2)=3$，得$f'_v(1,2)=-\frac{3}{2}$.
利用函数在$(1,2)$可微

$$f(x-\sin x+1,\sqrt{1+x^3}+1)-f(1,2)=f'_u(1,2)(x-\sin x)+f'_u(1,2)(\sqrt{1+x^3}+1-2)+o(\rho)$$

其中$\rho=\sqrt{(x-\sin x)^2+(\sqrt{1+x^3}-1)^2}$.
利用$x\to 0$时, $x-\sin x\sim\dfrac{1}{6}x^3$, $\sqrt{1+x^3}-1\sim\dfrac{1}{2}x^3$, 因此所求极限

$$\lim_{x\to 0}\frac{f(x-\sin x+1,\sqrt{1+x^3}+1)}{x^3}=\lim_{x\to 0}\frac{3(x-\sin x)-\frac{3}{2}(\sqrt{1+x^3}-1)+o(\rho)}{x^3}=-\frac{1}{4}.$$

七、 (本题共7 分) 设$f(x,y)\geq 0$ 在$D: x^2+y^2\leq 1$ 上有连续的一阶偏导数, 且在边界上取值为0, 证明

$$\left|\iint_D f(x,y)\mathrm{d}x\mathrm{d}y\right|\leq\frac{\pi}{3}\max_{(x,y)\in D}\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}.$$

证法1： 记$u(\rho,\theta)=f(\rho\cos\theta,\rho\sin\theta)$, $\quad M=\max\limits_{(x,y)\in D}\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}$.

边界条件$f\big|_{\rho=1}=0$, 对任意固定的θ, 由牛顿 - 莱布尼茨公式有

$$f(r\cos\theta,r\sin\theta)=f(r\cos\theta,r\sin\theta)-f(\cos\theta,\sin\theta)=-\int_r^1\frac{\partial f}{\partial\rho}\,d\rho.$$

$$\left|\frac{\partial f}{\partial\rho}\right|=|(f'_x,f'_y)\cdot(\cos\theta,\sin\theta)|\leq\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}$$

故 $$|f(r\cos\theta,r\sin\theta)|\leq\int_r^1\left|\frac{\partial f}{\partial\rho}\right|d\rho\leq\int_r^1 M\mathrm{d}\rho\leq M(1-r).$$

于是二重积分可估计为

$$\begin{aligned}\left|\iint_D f(x,y)\,dxdy\right|&=\left|\int_0^{2\pi}\int_0^1 f(r\cos\theta,r\sin\theta)\,r\,drd\theta\right|\\&\leq\int_0^{2\pi}\int_0^1 M(1-r)r\,drd\theta\leq\frac{\pi}{3}M.\end{aligned}$$

证法2： 由f在D上有连续一阶偏导数, 可知M有界. 对$\forall(x,y)\in D$, 由$(0,0)$到(x,y)引射线, 交圆周上点(x_0,y_0), 且f边界上取值为零, 故由中值定理和Cauchy不等式得

$$\begin{aligned}f(x,y)&=f(x_0,y_0)+f'_x(\xi,\eta)(x-x_0)+f'_y(\xi,\eta)(y-y_0)\\&=f'_x(\xi,\eta)(x-x_0)+f'_y(\xi,\eta)(y-y_0)\cdots\cdots(3\text{ 分})\\&\leq\sqrt{f'^2_x(\xi,\eta)+f'^2_y(\xi,\eta)}\sqrt{(x-x_0)^2+(y-y_0)^2}\leq M\left(1-\sqrt{x^2+y^2}\right)\end{aligned}$$

所以

$$\begin{aligned}\left|\iint_D f(x,y)\mathrm{d}x\mathrm{d}y\right|&\leq\iint_D|f(x,y)|\mathrm{d}x\mathrm{d}y\leq M\iint_D\left(1-\sqrt{x^2+y^2}\right)\mathrm{d}x\mathrm{d}y\\&=M\int_0^{2\pi}\int_0^1(1-r)r\,drd\theta\leq\frac{\pi}{3}M.\end{aligned}$$