

中国科学技术大学期中试卷  
2025-2026 学年第一学期

课程名称: 拓扑学(H)(英) . 课程编号: 001707  
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1. (5 points each) Which of the following statements are correct? It is enough to answer “True” or “False”. You don’t need to prove it.

- (1) A Mobius band is homeomorphic to itself twisted three times. True
- (2) A set is closed if and only if all of its points are limit points. False T (3) The composition of two continuous maps is continuous. True
- (4) Peano’s curves shows that  $[0, 1]$  is homeomorphic to  $[0, 1] \times [0, 1]$ . False
- (5) Any real-valued continous function defined on a compact subset of a metric space can be extended over the whole space. True
- (6) Any compact subset of  $\mathbb{R}$  must be a bounded closed interval. False
- (7)  $O(n)$  is connected. False
- (8) There exists a group  $G$  such that the Klein bottle is homeomorphic to  $\mathbb{R}^2/G$ . True

2. (20 points) Suppose that  $\mathbb{R}^n$  has the standard Euclidean topology, and  $X$  is an open subset of  $\mathbb{R}^n$  with the subspace topology. Suppose that  $Y$  is a path-connected topological space, and  $X$  is connected. Prove that  $X \times Y$  is path-connected.

First we prove that  $X$  is path-connected. (10 points)

(Proof 1) By a theorem on the textbook, any connected open subset of  $\mathbb{R}^n$  is path-connected.

(Proof 2) Fix  $x \in X$  and define  $A = \{y \in X, y \text{ can be path-connected to } x\}$ . Then  $A$  and  $X \setminus A$  are open. So  $X = A$ .

(Proof 3)  $X$  is locally path-connected. So connected implies path-connected.

Then we prove that  $X \times Y$  is path-connected. (10 points)

(Proof 1) For any  $(x_1, y_1), (x_2, y_2) \in X \times Y$ , if  $\gamma_1$  is a path from  $x_1$  to  $x_2$ ,  $\gamma_2$  is a path from  $y_1$  to  $y_2$ . Then  $\gamma(t) = (\gamma_1(t), \gamma_2(t))$  is a path from  $(x_1, y_1)$  to  $(x_2, y_2)$ .

(Proof 2) For any  $(x_1, y_1), (x_2, y_2) \in X \times Y$ , if  $\gamma_1$  is a path from  $x_1$  to  $x_2$ ,  $\gamma_2$  is a path from  $y_1$  to  $y_2$ . Then  $\gamma(t) = (\gamma_1(2t), y_1)$ ,  $t \in [0, \frac{1}{2}]$ ,  $\gamma(t) = (x_2, \gamma_2(2t - 1))$ ,  $t \in [\frac{1}{2}, 1]$  is a path from  $(x_1, y_1)$  to  $(x_2, y_2)$ .

3. (1) (5 points) Consider the space  $\mathbb{C}^2 \setminus \{0\}$  with the equivalent relationship  $\sim$  defined by  $x \sim y$  if and only if there exists  $\lambda \in \mathbb{C} \setminus \{0\}$  such that  $x = \lambda y$ . Prove that this is an equivalent relationship.

$x \sim x$  because  $x = 1 \cdot x$ . (1 point) If  $x \sim y$ , then  $x = \lambda \cdot y$ , so  $y = \frac{1}{\lambda}x$ , so  $y \sim x$ . (2 points) If  $x \sim y$ ,  $y \sim z$ , then  $x = \lambda y$ ,  $y = \mu z$ , so  $x = \lambda \mu z$ . (2 points)

(2) (10 points) Define  $X = (\mathbb{C}^2 \setminus \{0\}) / \sim$  with the identification topology, prove that  $X$  is Hausdorff.

(Proof 1) Suppose  $[x] \neq [y]$  are two points in  $X$ . Then  $x, y \in \mathbb{C}^2 \setminus \{0\}$ . Define  $U_x = \{z \in \mathbb{C}^2 \setminus \{0\}, |(z, x)| > (1 - \epsilon)|z||x|\}$ , and  $U_y = \{z \in \mathbb{C}^2 \setminus \{0\}, |(z, y)| > (1 - \epsilon)|z||y|\}$ . Then for sufficiently small  $\epsilon$ ,  $U_x \cap U_y = \emptyset$ . Let  $\pi$  be the map from  $\mathbb{C}^2 \setminus \{0\}$  to  $X$ . Then  $U_x = \pi^{-1}(\pi(U_x))$  is open. So  $\pi(U_x)$  is open. Similarly,  $\pi(U_y)$  is open. We also know that  $[x] \in \pi(U_x)$ ,  $[y] \in \pi(U_y)$ , and  $\pi(U_x) \cap \pi(U_y) = \emptyset$ . So  $X$  is Hausdorff.

(Proof 2) Suppose  $[x] \neq [y]$  are two points in  $X$ . If  $x, y \neq [(1, 0)]$ , then there exists  $a \neq b \in \mathbb{C}$  such that  $[x] = [(a, 1)]$ , and  $[y] = [(b, 1)]$ . Then we pick  $r > 0$  small such that  $B(a, r) \cap B(b, r) = \emptyset$ . Then  $U = \{(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}, z_1/z_2 \in B(a, r)\}$ , and  $V = \{(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}, z_1/z_2 \in B(b, r)\}$  are open,  $U = \pi^{-1}(\pi(U))$ , and  $V = \pi^{-1}(\pi(V))$ . So  $\pi(U)$  and  $\pi(V)$  are the required open sets. If  $x = [(1, 0)]$ , and  $y = [(b, 1)]$ , then we pick  $r > 0$  small such that  $\{z \in \mathbb{C}, |z| > \frac{1}{r}\} \cap B(b, r) = \emptyset$ . Then we choose  $U = \{(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}, |z_2/z_1| < r\}$  instead.

(3) (5 points) Define  $Y = \mathbb{C} \cup \{\infty\}$ . We define that  $O \subset Y$  is open if and only if either  $O$  is an open subset of  $\mathbb{C}$ , or  $O = (\mathbb{C} \setminus K) \cup \{\infty\}$  for a compact subset  $K$  of  $\mathbb{C}$ . Prove that this defines a topology on  $Y$ .

$\emptyset \in \mathcal{O}(Y)$  because  $\emptyset \in \mathcal{O}(\mathbb{C})$ . (1 point)

$Y \in \mathcal{O}(Y)$  because  $Y = (\mathbb{C} \setminus \emptyset) \cup \{\infty\}$ , and  $\emptyset$  is compact. (1 point)

For any  $O_i \in \mathcal{O}(Y), i \in I$ , if  $O_i$  are all in  $\mathcal{O}(\mathbb{C})$ , then  $\cup_i O_i \in \mathcal{O}(\mathbb{C})$ . If all of  $O_i = (\mathbb{C} \setminus K_i) \cup \{\infty\}$ , then  $\cup_i O_i = (\mathbb{C} \setminus \cap_i K_i) \cup \{\infty\}$ .  $K_i$  are closed and bounded, so their intersection is also closed and bounded, so compact. In the rest cases, we take the union of  $O_i \in \mathcal{O}(\mathbb{C})$  to get  $O \in \mathcal{O}(\mathbb{C})$ , we take the union of  $O_i = (\mathbb{C} \setminus K_i) \cup \{\infty\}$  to get  $(\mathbb{C} \setminus K) \cup \{\infty\}$ . Then  $\cup_i O_i = O \cup (\mathbb{C} \setminus K) \cup \{\infty\} = (\mathbb{C} \setminus (K \setminus O)) \cup \{\infty\}$ . Since  $O$  is open,  $\mathbb{C} \setminus O$  is closed, so  $K \setminus O = K \cap (\mathbb{C} \setminus O)$  is closed and bounded. (2 points)

For all  $U, V \in \mathcal{O}(Y)$ , if they are all in  $\mathcal{O}(\mathbb{C})$ , then their intersection is in  $\mathcal{O}(\mathbb{C})$ . If  $U \in \mathcal{O}(\mathbb{C})$ ,  $V = (\mathbb{C} \setminus K) \cup \{\infty\}$ , then  $U \cap V = U \cap (\mathbb{C} \setminus K) \in \mathcal{O}(\mathbb{C})$ . If  $U = (\mathbb{C} \setminus K_1) \cup \{\infty\}$ ,  $V = (\mathbb{C} \setminus K_2) \cup \{\infty\}$ , then  $U \cap V = (\mathbb{C} \setminus (K_1 \cup K_2)) \cup \{\infty\}$ .  $K_1 \cup K_2$  is also bounded and closed, so compact. (1 point)

(4) (10 points) Prove that  $Y$  is compact.

Suppose that  $Y = \cup_{i \in I} O_i$ , then there exists  $i_0$  such that  $\infty \in U_{i_0}$ . Then  $U_{i_0} = (\mathbb{C} \setminus K) \cup \{\infty\}$  for  $K$  compact. Then  $K$  is covered by  $O_i \cap \mathbb{C}$ . Since  $O_i \cap \mathbb{C} \in \mathcal{O}(\mathbb{C})$ , we see that there is a finite subcover  $K \subset \cup_{j=1}^n (O_{i_j} \cap \mathbb{C})$ . Then  $Y = \cup_{j=1}^n O_{i_j} \cup O_{i_0}$ .

(5) (10 points) There is a map  $f$  from  $Y$  to  $X$  defined by  $f(x) = [x, 1], x \in \mathbb{C}$ , and  $f(\infty) = [1, 0]$ , where  $[x, y]$  denotes the equivalent class of  $(x, y)$  under  $\sim$ . Prove that  $f$  is a homeomorphism from  $Y$  to  $X$ .

First,  $f$  is injective. (1 point)

Then  $f$  is surjective. (1 point)

Since  $Y$  is compact, and  $X$  is Hausdorff, it suffices to prove that  $f$  is continuous. (2 points)

$Y = \mathbb{C} \cup U$ , where  $U = (\mathbb{C} \setminus \{0\}) \cup \{\infty\}$ . The map  $f_1 : \mathbb{C} \rightarrow \mathbb{C}^2 \setminus \{0\}$  defined by  $f_1(z) = (z, 1)$

is continuous, so  $\pi \circ f_1$  is continuous. The map  $f_2 : U \rightarrow \mathbb{C}^2 \setminus \{0\}$  defined by  $f(z) = (1, \frac{1}{z})$ ,  $z \neq \infty$ , and  $f(\infty) = (1, 0)$  is also continuous, so  $\pi \circ f_2$  is also continuous. Since  $\mathbb{C}$  and  $U$  are all open in  $Y$ . By the glueing lemma, the glue of  $\pi \circ f_1$  with  $\pi \circ f_2$ , which is  $f$ , is continuous. (6 points)

There are many different proves of the continuity of  $f$ .