中国科学技术大学期末试卷
2025-2026 学年第一学期　A卷

课程名称: 拓扑学(H)(英) . 课程编号: 001707

考试时间: 2026年1月21日 2：30-4：30 考试形式: 闭卷

学生姓名: ______ 学　　号: ______

1.（4 points each）Which of the following statements are correct? It is enough to answer "True" or "False". You don't need to prove it.

(1) An arbitrary union of neighborhoods of x is also a neighborhood of x. True

(2) If $Y \subset X$, then $A \subset Y$ being closed in the subspace topology on Y implies that A is closed in the topology on X. False

(3) The continuous image of an open set is open. False

(4) Any bounded closed subset of a metric space is compact. False

(5) Let G be a connected topological group acting on X. Then X is connected if and only if X/G is connected. True

(6) The lens space is simply connected. False

(7) If $A \subset \mathbb{R}^2$ is homeomorphic to an interval, then $\mathbb{R}^2 \setminus A$ is connected. True

(8) There is no continuous surjective map from the disc to its boundary. False

(9) The connected sum of T^2 with $\mathbb{RP}^2$ is homeomorphic to the connected sum of three copies of $\mathbb{RP}^2$. True

(10) The Möbius band is triangulable. True

2. (20 points) Let $X = \{(x_1, ..., x_n) | x_i \geq 0, \sum_{i=1}^n x_i = 1\}$ be a simplex. Prove that any complex-valued continuous function defined on the boundary of X can be extended continuously to X.

(Method 1) By Tietze extension theorem, any real-valued function on a closed subset and be extended to a metric space. Now the boundary $\partial X = \{(x_1, ..., x_n) | x_i \geq 0, \sum_{i=1}^n x_i = 1, \exists i, s.t. x_i = 0\}$ is a closed subset of the metric space X. If $f = u + iv$ is the continous complex-valued function on ∂X, then both u and v are continous. Let $\bar{u}, \bar{v}$ be the extension, then $\bar{f} = \bar{u} + i\bar{v}$ is the extension to X.

(Method 2) Choose a point $x_0 \in X$. Then we define $f : X \setminus \{x_0\} \to \partial X$ be the intersection of the ray from x_0 to x with ∂X. Let $g : \partial X \to \mathbb{C}$ be the function. Then $h(x) = \frac{|x - x_0|}{|f(x) - x_0|} g(f(x))$ is continous outside x_0. When $x_i \to x_0$, then $\frac{|x_i - x_0|}{|f(x_i) - x_0|} \to 0$, and $|g(f(x))|$ is bounded because ∂X is compact. So $h(x_i)$ converges to x_0.

(Method 3) Let $\partial_i X = \{(x_1, ..., x_n) | x_k \geq 0, \sum_{k=1}^n x_k = 1, x_i = 0\}$. Let $d_i(x)$ be the distance to $\partial_i X$, and $f_i(x)$ be the point on $\partial_i X$ which is closest to x. Let $g : \partial X \to \mathbb{C}$ be the function.

Then $h(x) = \frac{\sum_{i=1}^n g(f_i(x))/d_i(x)}{\sum_{i=1}^n 1/d_i(x)}$ is continuous on X. (-5 points for minor issues such as missing the part from real to complex, or a wrong explanation of the continuity of h, -10 points for major issues such as a wrong usage of Tietze extension theorem)

3. (1) (5 points) Prove that for any three distinct points p_1, p_2, p_3 on the standard sphere S^2, the space $S^2 \setminus \{p_1, p_2, p_3\}$ is homeomorphic to $\mathbb{R}^2 \setminus \{(0,1), (0,-1)\}$.

By the stereographic projection, $S^2 \setminus \{p_1, p_2, p_3\}$ is homeomorphic to $\mathbb{R}^2 \setminus \{p_4, p_5\}$. Then the linear map provides a homeomorphism between $\mathbb{R}^2 \setminus \{p_4, p_5\}$ and $\mathbb{R}^2 \setminus \{(0,1), (0,-1)\}$. (-1 point if neither “stereographic projection” nor “linear” appear)

(2) (5 points) For three distinct points p_1, p_2, p_3 on the standard torus T^2, is the space $T^2 \setminus \{p_1, p_2, p_3\}$ homeomorphic to $\mathbb{R}^2 \setminus \{(0,1), (0,-1)\}$? A yes or no answer is sufficient; no proof or explanation is required.

No.

(3) (5 points) Define $U = (\mathbb{R} \times (-1, \infty)) \setminus \{(0,1)\}$, and $V = (\mathbb{R} \times (-\infty, 1)) \setminus \{(0,-1)\}$. Prove that there exists a topological group X such that U is homotopy equivalent to X.

(Method 1) $\mathbb{R} \times (-1, \infty)$ is homeomorphic to $\mathbb{C}$ using the map $(x, y) \to x + (\log(y+1))i$. So U is homeomorphic to $\mathbb{C}^*$. It is easy to see that the product on $\mathbb{C}^*$ makes it a topological group.

(Method 2.1) Choose S^1 be the circle with center $(0,1)$ and radius 1. For any point $x \in U$, the line segment between x to $(0,1)$ intersects S^1 at a unique point $f(x)$. Then $f : U \to S^1$ and the inclusion map $i : S^1 \to U$ satisfies $f \circ i = Id$, and $F(x,t) = tx + (1-t)f(x)$ provides a homotopy between the identity map and the map $i \circ f$. It is easy to see that the product on $S^1 \subset \mathbb{C}^*$ makes it a topological group.

(Method 2.2) Choose S^1 be the circle with center $(0,1)$ and radius 1. For any point $x \in U$, the line segment between x to $(0,1)$ intersects S^1 at a unique point $f(x)$. Define $F(x,t) = tx + (1-t)f(x)$, then $F(x,0) \in S^1$, $F(x,1) = x$, and $F(x,t) = x$ for all $x \in A$. So F is a deformation retraction from U to S^1. It is easy to see that the product on $S^1 \subset \mathbb{C}^*$ makes it a topological group. (-1 point for minor issues such as missing “$F(x,t) = x$ for all $x \in A$”, -2 points for answers like “It is obvious that U is homotopic to $\mathbb{R}^2 \setminus \{0\}$, and the latter deformatin retracts to S^1 without explanations”.)

(4) (5 points) Specify all covering spaces of X and their covering maps, where X is the topological group from part (3). You don’t need to prove it.

(Method 1) The universal cover $\exp : \mathbb{C} \to \mathbb{C}^*$, (3 points) and other covers $f_n : \mathbb{C}^* \to \mathbb{C}^*$ defined by $f_n(z) = z^n$ for all n. (2 points)

(Method 2) The universal cover $f : \mathbb{R} \to S^1$ defined by $f(x) = e^{ix}$, (3 points) and other covers $f_n : S^1 \to S^1$ defined by $f_n(z) = z^n$ for all n. (2 points)

(5) (10 points) Compute the fundamental group of $S^2 \setminus \{p_1, p_2, p_3\}$.

(Method 1) By (3), U, and V are homotopic to S^1. So their fundamental group are all isomorphic to $\mathbb{Z}$. Since $U \cap V$ is simply-connected, by van-Kampen’s theorem, $\pi_1(U \cup V, p) = \mathbb{Z} * \mathbb{Z}$.

(Method 2) There is a homotopic map between $S^2 \setminus \{p_1, p_2, p_3\}$ and the one-point union of

S^1 with S^1. So the fundamental group is $\mathbb{Z} * \mathbb{Z}$.

(-2 points for minor issues such as writing the free product as a product.)

(6) (10 points) There are two ways from China to Europe. The silk road economic belt is a land route by train, and the 21-st century maritime silk road is a sea route. There are currently three military conflict zones. Show that even if the starting point is allowed to vary within China and the destination within Europe, the land route and the sea route remain non-homotopic. Mathematically, let A and B be disjoint discs in $S^2 \setminus \{p_1, p_2, p_3\}$. Let γ_1, γ_2 be two paths from $p \in A$ to $q \in B$ such that $\gamma_1 \cdot \gamma_2^{-1}$ is non-trivial in the fundamental group. Prove that there is no continuous function $F : [0,1] \times [0,1] \to S^2 \setminus \{p_1, p_2, p_3\}$ such that $F(0,t) \in A$, $F(1,t) \in B$, $F(s,0) = \gamma_1(s)$, and $F(s,1) = \gamma_2(s)$.

(Method 1) By our assumption, there exist homeomorphisms $\varphi_A : A \to D$, $\varphi_B : B \to D$, where D is the standard disc. Define $G(s,t) = \varphi_A^{-1}((1-3s)\varphi_A(p) + 3s\varphi_A(F(0,t)))$, $s \in [0, 1/3]$, $G(s,t) = F(3s-1, t)$, $s \in [1/3, 2/3]$, $G(s,t) = \varphi_B^{-1}((3s-2)\varphi_B(q) + (3-3s)\varphi_B(F(1,t)))$, $s \in [2/3, 1]$, then G is continous, and it provides a homotopy between $e_p.\gamma_1.e_q$ with $e_p.\gamma_2.e_q$ rel $\{0,1\}$. We than multiply them with γ_2^{-1} to get a contradiction.

(Method 2) There exists a continous map $f, g : S^2 \setminus \{p_1, p_2, p_3\} \to S^2 \setminus \{p_1, p_2, p_3\}$, such that f maps A to a point, and B to another point, and both $f \circ g$ and $g \circ f$ are homotopic to the identity map. Then $f \circ F$ implies that $f_*(\gamma_1 \cdot \gamma_2^{-1})$ is trivial in the fundamental group. This is a contradiction because f_* is an isomorphism.

(No points with wrong method, such as a method which also works for $A = B = S^2 \setminus \{p_1, p_2, p_3\}$, -5 points for major mistakes, -2 for minor mistakes.)