2025 年随机过程期中试卷

注：原试卷为英文，试题主要由回忆得到，部分记号可能与原题不同

题目 1. 对于 σ 域 $\mathcal{F}$ 和随机变量 $X \in \mathcal{F}$，$E|X| < \infty$，考虑有界凸函数 $\varphi(x)$，若已知

$$E[\varphi(X)] = E[\varphi(E[X|\mathcal{F}])],$$

证明

$$E[\varphi(X)|\mathcal{F}] = \varphi(E[X|\mathcal{F}]).$$

（提示：你可以使用 Jensen 不等式）

题目 2. 考虑一列平方可积的随机变量 $\{X_n\}$，若 $\{S_n = \sum_{k=0}^n X_k\}$ 为关于一族 σ 域 $\{\mathcal{F}_n, n \geq 0\}$ 的鞅，证明

$$E[S_n^2] = E[\sum_{k=0}^n X_k^2].$$

题目 3. $\{X_n\}$ 为一列独立同分布的随机变量，满足 $P(X_n = -1) = \frac{1}{2}, P(X_n = 1) = \frac{1}{2}$；$\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ 为自然滤流；考虑简单对称随机游走 $S_n = \sum_{k=1}^n X_k, S_0 = 0$ 和停时 T 满足 $ET < \infty$，证明：

(i) $\{S_n^2 - n\}$ 是关于 $\mathcal{F}_n$ 的鞅；

(ii) $ET = E[S_T^2]$

(iii) $E[S_T] = 0$

题目 4. $\{X_n\}$ 为 Markov 链，转移矩阵为 $(p(x,y))$；记 $p^n(x,y) = P_x(X_n = y)$，$T_x = \inf\{k \geq 0; X_k = x\}$，$f^n(x,y) = P_x(T_y = n)$，证明

$$p^n(x,y) = \sum_{k=1}^n f^k(x,y)p^{n-k}(y,y).$$

题目 5. $\{X_n\}$ 是状态空间为 S 的 Markov 链，转移矩阵为 $(p(x,y))$；对 $a \in S$，记 $\tau_a = \inf\{k; X_k = a\}$；$\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ 为自然滤流；对于 $\alpha > 0$，满足以下条件的有界函数 h 被称作 α-上调和函数 (α-superharmonic function)

$$\sum_{y \in S} p(x,y)h(y) \leq e^{\alpha}h(x)$$

证明：(i) 对于 α-上调和函数 $h(y)$，$\{e^{-\alpha n}h(X_n)\}$ 是关于 $\mathcal{F}_n$ 的上鞅

(ii) 对于 α-上调和函数 $h(y)$，记 $p^n(x,y) = P_x(X_n = y)$，有

$$\sum_{y \in S} p^n(x,y)h(y) \leq e^{n\alpha}h(x)$$

(iii) $h(y) = E_y[e^{-\alpha\tau_a}]$ 为 α-上调和函数。

题目 6. $\{X_n\}$ 是状态空间为 S 的 Markov 链，转移矩阵为 $(p(x,y))$；给定其平稳分布 $\pi(y)$，证明：

(i)

$$\sum_{x\in S} p^n(x,y)\pi(x) = \pi(y)$$

(ii) 如果 p 不可约，则对任意的 $x \in S$ 有 $\pi(x) > 0$

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姓 名 ______________ 学 号 ______________ 学 院 ______________

题号	一	二	三	四	五	六	总分
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一、【10 分】

Let $\mathcal{F}$ be a σ-field. Let X be a bounded random variable and ψ a convex function. Prove that if $E[\psi(X)] = E\left[\psi\left(E[X|\mathcal{F}]\right)\right]$, then

$$E[\psi(X)|\mathcal{F}] = \psi\left(E[X|\mathcal{F}]\right).$$

(You may use Jensen's inequality.)

二、【15 分】

Let $\{X_n\}_{n\geq 0}$ be a sequence of square integrable random variables.

Suppose $\{S_n = \sum_{k=0}^{n} X_k\}_{n\geq 0}$ is a martingale w.r.t. a family of σ-fields $\mathcal{F}_n, n \geq 0$. Prove

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$$E[S_n^2] = E[\sum_{k=0}^{n} X_k^2].$$

三、【25 分】

Let $\{X_n\}_{n\geq 0}$ be i.i.d. random variables with $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$. $\{S_n\}_{n\geq 0}$ denotes the simple symmetric random walk, i.e. $S_n = X_1 + X_2 + \cdots + X_n$ with $S_0 = 0$. Introduce the σ-fields $\mathcal{F}_n = \sigma(X_1, ..., X_n)$ for $n \geq 1$. Let T be a stopping time with $E[T] < \infty$.

(i) Prove that $\{S_n^2 - n, n \geq 1\}$ is a martingale.

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(ii) Prove that $E[T] = E[S_T^2]$.

(iii) Prove that $E[S_T] = 0$.

四、【10 分】Let $(X_n, n \geq 0)$ be a Markov chain with state space S. Let $T_y = \inf\{m \geq 1; X_m = y\}$ and $f^k(x,y) = P_x(T_y = k)$. Recall $p^n(x,y) = P_x(X_n = y)$. Show that for $n \geq 1$,

$$p^n(x,y) = \sum_{k=1}^{n} f^k(x,y)p^{n-k}(y,y).$$

五、【25 分】

Let $(X_n, n \geq 0)$ be a Markov chain with state space S and the transition matrix $(p(x,y))$. For $a \in S$, define $\tau_a = \inf\{k \geq 0; X_k = a\}$. For $\alpha > 0$, we say that a bounded function h is α-superharmonic if for any $x \in S$,

$$\sum_y p(x,y)h(y) \leq e^{\alpha} h(x).$$

(i) If a function h is α-superharmonic, prove that under P_μ, $\{e^{-\alpha n} h(X_n)\}_{n\geq 0}$ is a supermartingale with respect to $\{\mathcal{F}_n = \sigma(X_k, k \leq n)\}_{n\geq 0}$, where μ is the distribution of X_0.

(ii) If a function h is α-superharmonic, prove

$$\sum_y p^n(x,y)h(y) \leq e^{\alpha n} h(x),$$

where $p^n(x,y) = P_x(X_n = y)$ is the n-steps transition probabilities.

(iii) Define $h(y) = E_y[e^{-\alpha \tau_a}]$ for $y \in S$. Prove that h is α-superharmonic.

六、【15 分】

Let $(X_n, n \geq 0)$ be a Markov chain with state space S and transition matrix $(p(x,y))$. Recall $p^n(x,y) = P_x(X_n = y)$. Let $\{\pi(x)\}$ be a stationary distribution.

(i) Prove

$$\sum_{x \in S} \pi(x) p^n(x,y) = \pi(y).$$

(ii) If the Markov chain is irreducible, prove that $\pi(y) > 0$ for $y \in S$.