2025 春随机过程 (研) 期末[1]

一、(15 分) Customers arrive at a post office according to a Poisson process $(N(t), t \geq 0)$ of rate $\lambda = 4$. When a customer arrives, with probability $\frac{1}{5}$ he receives service immediately. Let $\tilde{N}(t)$ denote the number of customers who arrive in the time interval $[0, t]$ and are serviced immediately.

(i) Explain why $\tilde{N}(t), t \geq 0$ has independent increments.

(ii) Determine the probability $\mathbb{P}(\tilde{N}(t) = k)$ for $t > 0, k \geq 1$.

二、(20 分) Let $\{B_t\}_{t\geq 0}$ be a Brownian motion. Show that for $\alpha > \frac{1}{2}$,

$$\lim_{n\to\infty} \frac{\sup_{n\leq t\leq n+1} B_t}{n^\alpha} = 0, \quad \text{a.s.}$$

(Hint: you may use the fact that $S_t = \sup\limits_{0\leq s\leq t} B_s$ has the same distribution as $|B_t|$) 注 Part of GTM274, Exercise 2.25.

三、(15 分) Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space.

(a) Let S be a stopping time and T an $\mathcal{F}_S$-measurable random variable with $S \leq T$. Prove that T is a stopping time.

(b) Let $\{X(t)\}_{t\geq 0}$ be an adapted, right continuous stochastic process. Let O be an open set. Define

$$T_O = \inf\{t \geq 0; X_t \in O\}$$

Prove that T_O is a stopping time w.r.t. $\{\mathcal{F}_{t+}\}_{t\geq 0}$.

注 GTM274, Proposition 3.9.

四、(10 分) Let $\{X_t\}_{t\in[0,\infty)}$ be an $(\mathcal{F}_t)$ submartingale. Show that for $T > 0$,

$$\sup_{0\leq t\leq T} \mathbb{E}[|X_t|] < \infty$$

注 GTM274, Proposition 3.13.

五、(20 分) Let $\{B_t\}_{t\geq 0}$ be an $(\mathcal{F}_t)$ Brownian motion started from 0. For $x \in \mathbb{R}$, let $T_x = \inf\{t \geq 0; B_t = x\}$. Fix two real numbers a and b with $a < 0 < b$ and set $T = T_a \wedge T_b$. Show that for $\lambda > 0$

$$\mathbb{E}\left[\exp(-\lambda T)\right] = \frac{\cosh\left(\frac{b+a}{2}\sqrt{\lambda}\right)}{\cosh\left(\frac{b-a}{2}\sqrt{\lambda}\right)}$$

注 GTM274, Exercise 3.27(1).

六、(20 分) Let E be a compact Polish space. Let $\{Q_t\}$ be a Feller semigroup on $C(E)$(sic, apparently referring to $C_0(E)$, similarly hereinafter) with the generator L. Denote by $(X_t, t \geq 0)$ the right continuous Markov process with the semigroup $\{Q_t\}$.

(i) Let $f \in D(L)$ with $h = Lf$. Show that under the probability measure $\mathbb{P}_x$,

$$M_t = f(X_t) - \int_0^t h(X_s)ds, \quad t \geq 0$$

is a martingale w.r.t. $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t), t \geq 0$ and deduce that for $s < t$,

$$\mathbb{E}_x\left[f(X_t) \mid \mathcal{F}_s\right] = f(X_s) + \mathbb{E}_x\left[\int_s^t h(X_u)du \mid \mathcal{F}_s\right]$$

(ii) Use (i) to prove that for any $f, g \in C(E)$,

$$\mathbb{E}_x\left[f(X_t)g(X_{t-})\right] = \mathbb{E}_x\left[f(X_{t-})g(X_{t-})\right]$$

注 GTM274, Exercise 6.27.

[1] KosmosX (aka *Spilopelia chinensis*) is acknowledged for documenting this exam paper.