Exam 2 for Differential Equations

May 24, 2025

ν always stands for the outward unit normal vector on the boundary, and U, Ω is always a bounded domain in $\mathbb{R}^n$.

1.(Sobolev space)

(a)(10 Marks) Suppose $|\log|x||^\alpha \in W^{1,2}(B_{\frac{1}{2}}(0)), B_{\frac{1}{2}}(0) \subseteq R^2$. Show the range of α.

(b)(10 Marks) Suppose $u \in W_0^{1,2}(U), U$ is an open bounded domain in R^n. Prove that $u^+ \in W_0^{1,2}(U)$.

(hint:$u^+ = \lim_{\varepsilon \to 0} F_\epsilon(u)$, $F_\epsilon(z) := \begin{cases} (z^2+\varepsilon^2)^{1/2} - \varepsilon \ , \ if \ z \geqslant 0 \\ 0 \quad if \ z < 0 \end{cases}$)

2.(Morrey inequality) Let $u \in C^1(R^n)$, prove the Morrey inequality in the following two steps:

(a)(10 Marks) $\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(x) - u(y)|dy \leq C(n)\int_{B(x,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}dy$

(b)(10 Marks) $[u]_{C^\alpha(R^n)} + |u|_{L^\infty(R^n)} \leq C(n)||u||_{W^{1,p}(R^n)}$, where $p > n$, $\alpha = 1 - \frac{n}{p}$.

3.(20 Marks) Consider the equation

$$\begin{cases} -\Delta u = f \ in \ U \subseteq R^n, f \in L^2(U) \\ u \in W_0^{1,2}(U) \end{cases}$$

Using the method of minimizing sequence to prove the existence(15 Marks) and uniqueness(5 Marks) of the weak solution.

4.(Fredholm alternative) Suppose $U = [0,1] \times [0,2]$.

(a)(10 marks) Consider the eigenvalue problem:

$$\begin{cases} \Delta u + \lambda u = 0 & in \quad U \\ u = 0 & on \quad \partial U \end{cases}$$

Find the eigenvalues λ_k and the corresponding eigenfunctions u_k.

(b)(10 marks) Consider the equation:

$$\begin{cases} \Delta u + \frac{5\pi^2}{4}u = 2x + y + a & in \quad U \\ u = 0 & on \quad \partial U \end{cases}$$

For which $a \in \mathbb{R}$ does this equation exist at least one solution ?

5.(a)(10 marks) Suppose $U \subseteq R^n$ a bounded domain,$U_T := U \times (0, T]$, $f \in C^1(U_T)$, $u \in C_1^2(U_T) \cap C(\overline{U_T})$solves the following equation: $\begin{cases} u_t = \triangle u + f(x,t) \ , in \ U_T \\ u|_{t=0} = g \\ u|_{\partial U} = \varphi \end{cases}$

Prove that$|u| \leqslant C \ in \ U_T$,where$C \sim |f|_{C^1}, |g|_{L^\infty}, |\varphi|_{L^\infty}$

(b)(10 marks) Suppose $R>0,\rho>0,f\in C^1\left(U_{R,\rho}\right),U_{R,\rho}=\left\{(x,t):|x|^2+|t|\leqslant(R+\rho)^2,t>0\right\}$ $U_R=\left\{(x,t):|x|^2+|t|\leqslant R^2,t>0\right\}$,$u\in C_1^2\left(U_{R,\rho}\right)\cap C\left(\overline{U_{R,\rho}}\right)$solves the following equation:

$$\begin{cases} u_t=\triangle u+f\left(x,t\right)\ \ ,in\ U_{R,\rho} \\ u|_{t=0}=g \\ u|_{\partial U}=0 \end{cases}$$

Prove that $\exists C\sim|f|_{C^1},|g|_{L^\infty},s.t.\,|Du|\leqslant C\ in\ U_R$
(hint:$\varphi=\xi^2\left|Du\right|^2+\alpha u^2+e^{\beta x_1}$,where $\xi=(R+\rho)^2-|x|^2-|t|$)

6.(a)(10 marks)(Moser iteration) Suppose $a_{ij}(x)\in L^\infty(B_R(0))$, $\lambda|\xi|^2\le a_{ij}(x)\xi_i\xi_j\le\Lambda|\xi|^2$. Consider the equation:

$$\begin{cases} -\sum\limits_{i,j}(a_{ij}(x)u_i)_j=0 & ,in\quad B_R(0) \\ u=0 & ,on\quad \partial B_R(0) \end{cases}$$

Assume $u>0$, $u\in C^\infty(B_R(0))$ is a weak solution of this equation. Prove that for any $p>1$, $\theta\in(0,1)$, we have:

$$\left(\int_{B_{\theta R}}u^{\frac{np}{n-2}}dx\right)^{\frac{n-2}{np}}\le C\left(\int_{B_R}u^p dx\right)^{\frac{1}{p}}$$

and give the expression of C.

(b)(10 marks) Suppose $a_{ij}(x)\in C^1(\overline{\Omega})$, , $f\in L^2(\Omega)$, $\lambda|\xi|^2\le a_{ij}(x)\xi_i\xi_j\le\Lambda|\xi|^2$. Consider the equation:

$$\begin{cases} -\sum\limits_{i,j}(a_{ij}(x)u_i)_j=f & ,in\quad \Omega \\ u=0 & ,on\quad \partial\Omega \end{cases}$$

Assume $u\in H_0^1(\Omega)$ is a weak solution of this equation. For any subset $V\subset\subset\Omega$, prove that: $u\in H^2(V)$ and

$$\int_V|D^2u|^2dx\le C\int_\Omega(f^2+u^2)dx$$

where $C\sim V,\Omega$, coefficients of L.