2025年春季学期 现代偏微分方程 第一次测验

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First Exam for PDE

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ν always stands for the unit outward normal vector on the boundary, and U is always a bounded C^1 domain in $\mathbb{R}^n$.

1.(Sobolev space)

(a)(10 Marks) Suppose $B_1(0) \subset \mathbb{R}^n$, for what values of α does $|x|^\alpha \in W^{1,2}(B_1(0))$?

(b)(10 Marks) Denote $\Omega = \{(x,y) : 0 < x < 1, 0 < y < x^2\}$, $p > 6$. Then for what values of α does $v(x,y) = x^\alpha \in W^{1,2}(\Omega) \backslash L^p(\Omega)$?

2.(Integration by parts)

(a)(10 Marks) Consider

$$\Delta u = f \quad a.e.\ in\ U$$

where $u \in W^{2,2}(U)$, $f \in L^2(U)$. Prove that: for any $V \Subset U$, we have $\int_V |\nabla u|^2 dx \leq C \int_U (u^2 + f^2) dx$, where C depends only on $n, dist(V, \partial U)$.

Hint: Multiply $u\xi^2$.

(b)(10 Marks) Suppose $\alpha \in \mathbb{R}$ and $u \in C^2(\mathbb{R}^n)$ be a positive solution to $\Delta u + u^\alpha = 0$ in $\mathbb{R}^n$. Denote $\eta \in C_0^\infty(B_{2R}(0))$ such that $\eta = 1$ in $B_R(0)$, $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq \frac{C_n}{R}$. For any $a \in \mathbb{R}$, $p > 4$, prove that

$$\int u^{a+2\alpha} \eta^p dx \leq C_1 \int u^{a-2} |\nabla u|^4 \eta^p dx + \frac{C_2}{R^4} \int u^{a+2} \eta^{p-4} dx$$

where C_1, C_2 depends only on n, a, α, p.

Hint: Multiply $u^{a+\alpha}\eta^p$.

3.(Properties of harmonic function)

Suppose $u \in C^\infty(B_R)$ is a solution to $\Delta u = 0\ in\ B_R \subseteq R^n$.

(a)(10 Marks) Prove that $\sup\limits_{B_{\frac{R}{2}}} |Du| \leq \frac{C_n}{R} \sup\limits_{B_R} |u|$.

(b)(10 Marks) Assume additionally $u > 0$ in B_R, prove that $\sup\limits_{B_{\frac{R}{2}}} |\nabla \log u| \leq \frac{C_n}{R}$.

(c)(10 Marks) Consider equation

$$\begin{cases} \Delta u(x,y) = xy\ in\ B_1(0) \subseteq \mathbb{R}^2 \\ u = 0\ on\ \partial B_1(0) \end{cases}$$

. Compute $u(0)$.

4.(Strong maximum principle)

(a)(5 Marks) Suppose $u \in C^2(U) \cap C^1(\bar{U})$ is a solution to

$$\begin{cases} \Delta u = -2\ in\ U \\ u = 0\ on\ \partial U \end{cases}$$

Prove that $u > 0$ in U and $\frac{\partial u}{\partial \nu} < 0$ on ∂U.

(b)(10 Marks) Suppose $\Omega \subseteq \mathbb{R}^2$ is a connected domain and $u \in C^4(\Omega)$ satisfies $\Delta u = f > 0$ in $\Omega \subseteq \mathbb{R}^2$. If $f \in C^2(\Omega)$ and $\frac{1}{f(x)}$ is convex, prove that $Rank(D^2 u) \equiv Constant$ in Ω.

5.(C^0 estimate)

(a)(10 Marks) Suppose $u \in C^2(U) \cap C(\bar{U})$ is a solution to

$$\begin{cases} \Delta u = f \ in \ U \subseteq R^n \\ u = \phi \ on \ \partial U \end{cases}$$

where $f \in C(\bar{U})$, $\phi \in C(\partial U)$. Prove that $\sup\limits_{\bar{U}} |u| \leq \sup\limits_{\partial U} |\phi| + \frac{d^2}{2n} \sup\limits_{\bar{U}} |f|$, where $d = diam(U)$.

(b)(5 Marks) Suppose $u \in C^2(U) \cap C^1(\bar{U})$ is a solution to

$$\begin{cases} \Delta u = 0 \ in \ U \\ \frac{\partial u}{\partial \nu} = -u + \phi \ on \ \partial U \end{cases}$$

where $\phi \in C(\partial U)$. Prove $\sup\limits_{\bar{U}} |u| \leq \sup\limits_{\partial U} |\phi|$.

(c)(10 Marks) Suppose $u \in C^2(U) \cap C^1(\bar{U})$ is a solution to

$$\begin{cases} \Delta u = f \ in \ U \\ \frac{\partial u}{\partial \nu} = -u + \phi \ on \ \partial U \end{cases}$$

where $f \in C(\bar{U})$, $\phi \in C(\partial U)$. Prove $\sup\limits_{\bar{U}} |u| \leq \sup\limits_{\partial U} |\phi| + \frac{d^2}{2n} \sup\limits_{\bar{U}} |f|$, where $d = diam(U)$.

6.(C^1 estimate)

Remark: Select one question from (b) and (c) to answer!!

Suppose $u \in C^3(U) \cap C^1(\bar{U})$ is a solution to

$$\begin{cases} \Delta u = f \ in \ U \\ u = 0 \ on \ \partial U \end{cases}$$

where $f \in C^1(\bar{U})$.

(a)(10 Marks) Prove that

$$\sup_{\bar{U}} |\nabla u| \leq C(1 + \sup_{\partial U} |\nabla u|),$$

where C depends only on $n, |f|_{C^1(\bar{U})}, diam(U)$.

Hint: Consider $\varphi = |\nabla u|^2 + \alpha u^2 + \beta |x|^2$.

(b)(Optional:10 Marks) For any $V \Subset U$, prove that

$$\sup_{\bar{V}} |\nabla u| \leq C,$$

where C depends only on $n, |f|_{C^1(\bar{U})}, dist(V, \partial U)$.

Hint: Consider $\varphi = \xi^2 |\nabla u|^2 + \alpha u^2 + \beta |x|^2$.

(c)(Optional:10 Marks) Assume additionally that U satisfies the uniform exterior sphere property. Prove that $\sup\limits_{\partial U} |\nabla u| \leq C$, where C depends only on $n, |f|_{C^1(\bar{U})}, U$.