Problem 1 (3 points each, 30 points in total)

Reading comprehension. Here is a part from our course notes:

定理 2.9.6. 【LCH空间的Tietze扩张定理】
设 X 为$\boxed{\text{LCH空间}}$, K 为X的紧子集，则任意$\boxed{\text{连续函数}}$ $f: K \to [-1,1]$ 都可被扩张为具有紧支集的连续函数 $\tilde{f}: X \to [-1,1]$.

证明　证明跟定理2.8.9非常类似，即$\boxed{\text{取开集}V\text{使得}\overline{V}\text{是紧集，且}K \subset V \subset \overline{V} \subset X}$ 然后对子空间$\overline{V}$[redacted]应用Tietze扩张定理：$\boxed{K \cup (\overline{V} \setminus V)\text{是}\overline{V}\text{中的闭集，}}$ 函数 $f_1(x) = \begin{cases} f(x), & x \in K \\ 0, & x \in \overline{V} \setminus V \end{cases}$ 是定义在该闭集上的连续函数，从而可以被扩张为连续函数$\tilde{f}_1 : \overline{V} \to [-1,1]$. 最后将$\tilde{f}_1$做$\boxed{\text{零扩张}}$得到函数$\tilde{f}: X \to [-1,1]$. 由粘接引理，$\tilde{f}$是连续函数，而且$\boxed{\text{supp}(\tilde{f})} \subset \overline{V}$是[redacted]，从而也是紧集. □

(1) (5×3′) Write down the definitions of $\boxed{\text{LCH空间}}$, $\boxed{\text{连续映射}}$, $\boxed{\text{零扩张}}$, $\boxed{\text{supp}(f)}$.

(2) (1×3′) Write down the full statement of the usual Tietze扩张定理.

(3) (1×3′) Write down the 粘接引理 that we used above.

(4) (1×3′) Under what condition, we may $\boxed{\text{取开集}V\text{使得}\overline{V}\text{是紧集，且}K \subset V \subset \overline{V} \subset X}$?

(5) (1×3′) Explain why: $\boxed{K \cup (\overline{V} \setminus V)\text{是}\overline{V}\text{中的闭集}}$.

(6) (2×3′) Two sentences were painted black with ink by the DAMN PROFESSOR ROCKET who teach you topology. The first one explains why we can apply Tietze extension theorem in $\overline{V}$, and the second one explains why supp(f) is compact. Recover the reason.

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(1)• $\boxed{\text{LCH空间}}$ means Locally compact and Hausdorff, where

- X is Locally compact means

- X is Hausdorff means

• $f: X \to Y$ is a $\boxed{\text{连续映射}}$ means

(Answers to Problem 1 – continued)

- For $A \subset X$, the $\boxed{\text{零扩张}}$ of $f : A \to \mathbb{R}$ means

- Given $f : X \to \mathbb{R}$, $\boxed{\mathrm{supp}(f)} =$

(2) Tietze扩张定理:

(3) There are many different 粘接引理. What we used is:

(4) The condition under which we can $\boxed{\text{取开集}V\text{使得}\overline{V}\text{是紧集，且}K \subset V \subset \overline{V} \subset X}$

(5) $\boxed{K \cup (\overline{V} \setminus V)\text{是}\overline{V}\text{中的闭集}}$ because:

(6) - We can't apply Tietze extension theorem in X directly, but we can apply Tietze extension theorem in $\overline{V}$, because:

- $\mathrm{supp}(f)$ is compact because:

Problem 2 (2 points each, 20 points in total)

Which of the following statements are correct? Write a "T" before each correct statement, and write an "F" before each wrong statement.

( ) If $f : [a,b] \to \mathbb{R}$ is continuous, and $f([a,b]) \supset [a,b]$, then there exists $p \in [a,b]$ such that $f(p) = p$.

( ) For any subset $A \subset (X,d)$, define $d_A(x) = \inf_{a\in A} d(x,a)$. Then $d_A(x) = 0$ if and only if $x \in A$.

( ) Let $\mathcal{B}$ be a basis of a topological space $(X, \mathscr{T})$, then $\mathcal{B}(x) = \{B \in \mathcal{B} \mid x \in B\}$ is a neighborhood basis at x.

For any subset A in a metric space (X,d), the derived set A' must be closed.

( ) If A, B are compact subsets of $(X, \mathscr{T})$, then $A \cap B$ is compact.

For the Cantor set C, we have $C' = C$.

Any non-compact metric space admits a completion.

The set $\mathbb{Q}$ (endow with the standard metric topology) is locally compact.

The product of 2024 σ-compact spaces is still σ-compact.

( ) If f is a continuous function on $[0,1]$ and $\int_0^1 f(x)e^{3kx}dx = 0$ holds for all $k \in \mathbb{N}$, then $f \equiv 0$.

( ) Any subspace of a (T4) space is still a (T4) space.

Problem 3 (20 points)

Consider the following topologies on $\mathbb{R}^2$:

$\mathscr{T}_1 =$ the cofinite topology

$\mathscr{T}_2 =$ the topology generated by the metric $d((x_1, x_2), (y_1, y_2)) = \min\{1, |x_1 - y_1| + |x_2 - y_2|\}$

$\mathscr{T}_3 =$ the topology generated by the basis $\{[m, m+1) \times [n, n+1) \mid m, n \in \mathbb{Z}\}$

$\mathscr{T}_4 =$ the topology generated by the sub-basis $\{l \mid l \text{ is a line in } \mathbb{R}^2\}$

(1) Show that $(\mathbb{R}^2, \mathscr{T}_1)$ is compact.

(2) $(\mathbb{R}^2, \mathscr{T}_2)$ is second countable. Write down a countable basis (No proof is needed).

(3) Is $(\mathbb{R}^2, \mathscr{T}_3)$ (T2)? (T3)? (T4)? (A1)? Separable? (No proof is needed).

(4) $\mathscr{T}_4$ is a topology that we are familiar with. What is it? Explain why.

Problem 4 (20分)

Consider the topology $\mathscr{T}$ on $\mathbb{R}$ given by

$$\mathscr{T} = \{\emptyset, \mathbb{R}\} \cup \{(a, +\infty), [a, \infty) \mid a \in \mathbb{R}\}.$$

(1) What is the closure of $(0, 1)$? Prove your conclusion.

(2) Prove: If $f : \mathbb{R} \to \mathbb{R}$ is increasing 【i.e. $a < b \Rightarrow f(a) \leq f(b)$】, then $f : (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T})$ is continuous.

(3) Prove: If $f : (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T})$ is continuous, then $f : \mathbb{R} \to \mathbb{R}$ is increasing

(4) Let $\mathscr{T}_0$ the usual topology on $\mathbb{R}$. Find all continuous maps $f : (\mathbb{R}, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_0)$ and prove your conclusion.

Problem 5 (15 points)

Let $f : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be maps between topological spaces. Define

$$f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2, \quad (f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2)).$$

Endow the product spaces with the product topology. Prove:

(1) If f_1, f_2 are continuous, so is $f_1 \times f_2$.

(2) If f_1, f_2 are open maps, so is $f_1 \times f_2$.

(3) Find an example (with explanation) so that f_1, f_2 are closed, but $f_1 \times f_2$ is not closed.

Problem 6 (25分)

(1) Show that if (X, d) is a locally compact metric space, then for any $x \in X$, there exists $r > 0$ so that the closed ball $\overline{B(x, r)}$ is compact.

(2) We learned in class that any compact metric space is complete. But convergence is a local property. So it is natural to write and prove the following "generalization":

Theorem: Any locally compact metric space (X, d) is complete.

证明. Let (x_n) be any Cauchy sequence in (X, d). By definition of Cauchy sequence, for n, m large enough, the distance $d(x_n, x_m)$ can be as small as we want. So we take $r > 0$ so that $\overline{B(x_n, r)}$ is compact, and $d(x_n, x_m) < r$ for $m > n$. Since $\overline{B(x_n, r)}$ is compact, $(x_m)_{m>n}$ has a subsequence that converges to $a \in \overline{B(x_n, r)}$. So (x_n) has a convergent subsequence, i.e. X is complete. □

Explain the mistake in the proof.

(3) Write down (with explaination) a locally compact metric space (X, d) that is not complete.

(4) Suppose (X, d) is a locally compact metric space, such that for any $x, y \in X$ there is an isometry $f : (X, d) \to (X, d)$ so that $f(x) = y$. Prove: (X, d) is complete.

(5) Let (X, d) be a locally compact metric space, so that any closed ball $\overline{B(x, r)}$ is compact. Prove: The isometry group $\mathrm{Isom}(X, d)$ is locally compact in $(\mathcal{C}(X, X), \mathcal{T}_{c.o.})$.

Problem 7 (10 points)

On $\mathbb{R}^n$, consider the Euclidean metric d_0 and the l^∞ metric d_∞.

(i.e. $d_\infty((x_i),(y_i)) = \sup_{1\leq i\leq n} \{|x_i - y_i|\}$)

(1) On the set $X = \{u, s, t, c\}$, consider the metric d given by

$$d(u,c) = d(s,c) = d(t,c) = 1, \quad d(u,s) = d(u,t) = d(s,t) = 2.$$

Show that there is no isometric embedding $f : (X, d) \to (\mathbb{R}^n, d_0)$.

(2) Let (Y, d) be a metric space that consists of exactly n points. Prove: (Y, d) can be isometrically embedded into $(\mathbb{R}^n, d_\infty)$.

Problem 8 (10 points)

Let X be compact Hausdorff topological space, and $f : X \to X$ be a continuous map. Prove: there exsits a closed subset $A \subset X$ so that $f(A) = A$.

[It seems to me that part of this problem is harder than others.]

拓扑学 (H) 期中试卷简析, 2024 秋

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期中试卷

1 (1) • locally compact: $\forall x \in X, \exists$ 紧邻域 $K \in \mathscr{N}(x)$, 则称 X 局部紧.
Hausdorff: $\forall x \neq y \in X, \exists U, V \in \mathscr{T}_X, x \in U, y \in V, U \cap V = \varnothing$, 则称 X Hausdorff.

• 连续映射: $\forall V \in \mathscr{T}_Y, f^{-1}(V) \in \mathscr{T}_X$, 则称 f 是连续映射.

• 零扩张: $f\colon A \to \mathbb{R}$ 的零扩张是指函数 $g\colon X \to \mathbb{R}, x \mapsto \begin{cases} f(x), & x \in A, \\ 0, & x \in A^c. \end{cases}$

• $\operatorname{supp} f$: 对于函数 $f\colon X \to \mathbb{R}, \operatorname{supp} f = \overline{f^{-1}(\mathbb{R} - \{0\})}$.

(2) Tietze 扩张定理 (定理 2.9.4.): 拓扑空间 $(X, \mathscr{T})$ 是 (T4) 空间当且仅当对于任意闭集 $A \subset X, A$ 上的任意连续函数 $f\colon A \to [-1, 1]$ 可以被扩张为 X 上的连续函数 $\widetilde{f}\colon X \to [-1, 1]$.

(3) 粘贴引理: 设 $X = A \cup B, A, B$ 闭集, 则函数 $f\colon X \to \mathbb{R}$ 连续 $\iff$ $f|_A\colon A \to \mathbb{R}$ 连续且 $f|_B\colon B \to \mathbb{R}$ 连续.

(4) 取开集 V 使得 $\overline{V}$ 是紧集, 且 $K \subset V \subset \overline{V} \subset X$: X 是 LCH 空间.

(5) $K \cup (\overline{V} - V)$ 是 $\overline{V}$ 中的闭集: $K \subset \overline{V} \subset X$ 是 Hausdorff 空间的紧子集, 则为闭集, 故 $K \cup (\overline{V} - V) = (K \cup V^c) \cap \overline{V}$ 是 $\overline{V}$ 中的闭集.

(6) (P_{158}) 因为 LCH 空间未必 (T4) , 但 $\overline{V}$ CH $\implies$ (T4).
$\operatorname{supp} f \subset \overline{V}$ 是紧空间的闭子集.

2 TFTTFTTFTTF

3 (1) $\forall \mathbb{R}^2 = \bigcup_\alpha U_\alpha, \varnothing \neq \mathscr{U} = \{U_\alpha\} \subset \mathscr{T}_1$, 取 $U_0 \in \mathscr{U}$, 则 $U_0^c = \{x_1, \cdots, x_n\} \subset \bigcup\limits_{\alpha \neq 0} U_\alpha \implies$

$\exists U_1, \cdots, U_n$ s.t. $x_i \in U_i, 1 \leqslant i \leqslant n \implies \mathbb{R}^2 = \bigcup\limits_{i=0}^{n} U_i$.

(2) $\{B_d((p_1, p_2), q)\colon p_1, p_2, q \in \mathbb{Q}\}$.

(3) (T3)(T4)(A1) separable ✓, (T2) ×.

(4) $\mathscr{T}_4 = \mathscr{T}_{\text{discrete}}$. 因为 $\forall (u, v) \in \mathbb{R}^2, \{(u, v)\} = (\{u\} \times \mathbb{R}) \cap (\mathbb{R} \times \{v\}) \in \mathscr{T}_4 \implies \mathscr{T}_4 = 2^{\mathbb{R}^2}$.

4 (1) $(-\infty,1)$: 闭集形如 $\varnothing,\mathbb{R},(-\infty,a]$ 或 $(-\infty,a)$, 且 $(-\infty,a]\supset(0,1)\iff a>1;(-\infty,a)\supset(0,1)\iff a\geqslant 1$, 则 $\overline{(0,1)}=\bigcap\limits_{a>1}(-\infty,a]\cap\bigcap\limits_{a\geqslant 1}(-\infty,a)=(-\infty,1)$.

(2) $\forall a\in\mathbb{R},\forall x\in f^{-1}(a,+\infty),[x,+\infty)\subset f^{-1}(a,+\infty);\forall b\in\mathbb{R},\forall y\in f^{-1}[b,+\infty),[y,+\infty)\subset f^{-1}[b,+\infty)\implies f^{-1}(a,+\infty),f^{-1}[b,+\infty)\in\mathscr{T},\forall a,b\in\mathbb{R}$ 即 f 连续.

(3) 否则设 $f(x)>f(y),x<y$, 注意到 $\forall t,[t,+\infty)$ 是包含 t 的最小开集, 则 $x\in f^{-1}(f(y),+\infty)\implies[x,+\infty)\subset f^{-1}(f(y),+\infty)$, 但 $y\notin f^{-1}(f(y),+\infty)$, 矛盾.

(4) f 连续 $\iff\forall x,f(x)\in(a,b),f([x,+\infty))\subset(a,b)\iff f$ 常值函数.

5 (1) 由定理 1.83, 我们只需对 $Y_1\times Y_2$ 的一个拓扑基 $\{V_1\times V_2\colon V_1\in\mathscr{T}_{Y_1},V_2\in\mathscr{T}_{Y_2}\}$ 验证开集的原像开. ······ (1 分)

这是因为 f_1,f_2 连续 $\implies f_1^{-1}(V_1)\in\mathscr{T}_{X_1},f_2^{-1}(V_2)\in\mathscr{T}_{X_2}\implies(f_1\times f_2)^{-1}(V_1\times V_2)=f_1^{-1}(V_1)\times f_2^{-1}(V_2)\in\mathscr{T}_{X_1\times X_2}$. ······ (4 分)

(2) $\forall U\in\mathscr{T}_{X_1\times X_2},\forall y\in f(U)$, 取 $x=(x_1,x_2)\in(f_1\times f_2)^{-1}(y)\cap U$, 则 $\exists x_1\in U_1\in\mathscr{T}_{X_1},x_2\in U_2\in\mathscr{T}_{X_2}$ s.t. $x\in U_1\times U_2\subset U\implies y=(f_1\times f_2)(x)=(f_1(x_1),f_2(x_2))\in f_1(U_1)\times f_2(U_2)\subset f(U)$, 且 f_1,f_2 为开映射 $\implies f_1(U_1)\in\mathscr{T}_{Y_1},f_2(U_2)\in\mathscr{T}_{Y_2}\implies f_1(U_1)\times f_2(U_2)\in\mathscr{T}_{Y_1\times Y_2}$. 故由 $y\in f(U)$ 的任意性, $f(U)\in\mathscr{T}_{Y_1\times Y_2}$. 故由 $U\in\mathscr{T}_{X_1\times X_2}$ 的任意性, $f_1\times f_2$ 为开映射. ······ (5 分)

注记. 不加证明地声称只需对 $X_1\times X_2$ 的一个拓扑基 $\{U_1\times U_2\colon U_1\in\mathscr{T}_{X_1},U_2\in\mathscr{T}_{X_2}\}$ 验证开映射扣 2 分.

(3) 模仿 $\mathbb{R}^2\twoheadrightarrow\mathbb{R}$ 的投影映射: 考虑 $f_1=\mathrm{id}_{\mathbb{R}}\colon\mathbb{R}\to\mathbb{R},f_2\colon\mathbb{R}\to\{0\}$, 则 $f_1\times f_2\colon\mathbb{R}^2\twoheadrightarrow\mathbb{R}\times\{0\},(x,y)\mapsto(x,0)$ 将闭集 $\{(x,y)\colon xy=1\}$ 映至非闭集 $(\mathbb{R}-\{0\})\times\{0\}$. (5 分)

6 (1) X 局部紧 $\implies\forall x\in X,\exists x\in U\subset K$ 紧, $U\in\mathscr{T}_X$, ······ (2 分)

则因为度量拓扑由开球生成, $\exists r>0$ s.t. $B(x,2r)\subset U$, ······ (2 分)

则 $\overline{B(x,r)}\subset K$ 为紧空间的闭子集, 故紧. ······ (1 分)

注记. (i) 本题中 $\overline{B(x,r)}$ 中指闭球 $\{y\in X\colon d(y,x)\leqslant r\}$. 由于到一点距离 $f(y)=d(y,x)$ 的连续性 $\left(f\left(\overline{A}\right)\subset\overline{f(A)}\right)$, 对于开球的闭包 $\overline{(B(x,r))}$, 我们有 $\left\{d(y,x)\colon y\in\overline{(B(x,r))}\right\}\subset\overline{\{d(y,x)\colon y\in B(x,r)\}}\subset[0,r]\implies\overline{(B(x,r))}\subset\overline{B(x,r)}$. 本次考试本题中混淆这两个概念不扣分.

但一般情况下这可能是真包含, 例如: 在离散度量空间 $\{0,1\}$ 中, $B(0,1)=\varnothing\implies\overline{(B(0,1))}=\varnothing\subsetneq\{1\}=\overline{B(0,1)}$.

我们观察到这里 $\{d(y,x)\colon y\in B(x,r)\}\subset[0,r)$ 也是真包含. 然而即使 $\{d(y,x)\colon y\in B(x,r)\}=[0,r)$ 也不一定有 $\overline{(B(x,r))}=\overline{B(x,r)}$, 例如: 在 $\{-1\}\cup[0,1]$ 中, $d(x,0)=x(\forall x\in[0,1))$, 但 $\overline{(B(0,1))}=[0,1]\subsetneq\{-1\}\cup[0,1]=\overline{B(0,1)}$.

我个人习惯使用 $\overline{B}(x,r)$ 表示闭球而使用 $\overline{B(x,r)}$ 表示开球的闭包. 作业与考试中无论任何自用记号都请预先注明!

(ii) 见 PSet 5-1(4) 后的注记, 根据 $B(x,r)\subset U\subset K$ 未使用度量空间的 Hausdorff 性质直接得到 $\overline{(B(x,r))}\subset K$ 扣 1 分.

(2) 不一定存在 (n,r) 使得 $\overline{B(x_n,r)}$ 紧且 $\forall m>n, d(x_n,x_m)<r$. ⋯⋯(5 分)

注记. 例如观察 $\left\{\frac{1}{n}\right\}_{n>1}\subset(0,1)$.

(3) $(0,1)$. ⋯⋯(3 分)

解释: 局部欧氏故局部紧, 但为 $[0,1]$ 的非闭子空间. ⋯⋯(2 分)

注记. 言之有理即可. 特别地, 根据 PSet6-2(2)(b)(i), $\mathbb{Q}$ 不局部紧.

(4) $\forall\{x_n\}$ Cauchy 列. 设 $r>0$ 使 $\overline{B(x_1,r)}$ 紧, ⋯⋯(2 分)

且 $\forall m,n\geqslant N, d(x_n,x_m)<r$,

取等距同构 $f\colon(X,d)\to(X,d)$ s.t. $f(x_1)=x_N$, 则 $\overline{B(x_N,r)}=f\left(\overline{B(x_1,r)}\right)$ 紧, (2 分)

故 $\overline{B(x_N,r)}$ 完备 $\implies$ $\{x_n\}_{n\geqslant N}\subset\overline{B(x_N,r)}$ 在 $\overline{B(x_N,r)}$ 中收敛 $\iff$ 在 X 中收敛.

故由 Cauchy 列 $\{x_n\}$ 的任意性, X 完备. ⋯⋯(1 分)

(5) X 度量空间 $\implies$ 由命题 2.4.21, $\mathscr{T}_{\text{c.o.}}=\mathscr{T}_{\text{c.c.}}$. ⋯⋯(1 分)

$\forall f\in\mathrm{Isom}(X,d),\forall x_0\in X$, 我们证明 f 的邻域 $\mathscr{F}:=B(f;\{x_0\},\varepsilon)$ 预紧.

$\forall g\in\mathscr{F}, x,y\in X, d(g(x),g(y))=d(x,y)$, 即 $\mathscr{F}$ (逐点) 等度连续. ⋯⋯(2 分)

$d(g(x),f(x_0))\leqslant d(g(x),g(x_0))+d(g(x_0),f(x_0))<d(x,x_0)+\varepsilon\implies\mathscr{F}_x\subset B(f(x_0),d(x,x_0)+\varepsilon)$ 逐点预紧, 故由定理 2.5.8, $\mathscr{F}$ 预紧. ⋯⋯(2 分)

7 (1) 反证法. $d_0(f(u),f(c))+d_0(f(s),f(c))=d_0(f(u),f(s))\implies f(c)$ 恰为 $f(u),f(s)$ 中点, 同理 $f(c)$ 也为 $f(u),f(t)$ 中点, 矛盾! ⋯⋯(5 分)

(2) 类似定理 2.7.11 的证明: 考虑 $f_j(x_i)=d(x_i,x_j)$. ⋯⋯(3 分)

则 $d_\infty(f(x_i),f(x_j))=\max\limits_{1\leqslant k\leqslant n}|f_k(x_i)-f_k(x_j)|=\max\limits_{1\leqslant k\leqslant n}|d(x_i,x_k)-d(x_j,x_k)|$. 由 $|\cdot|$ 的三角不等式, $d_\infty(f(x_i),f(x_j))=d(x_i,x_j)$, ⋯⋯(1 分)

且 $k=i$ 或 j 时等号成立, 故 $d_\infty(f(x_i),f(x_j))\leqslant d(x_i,x_j)$, 即 f 是等距嵌入. ⋯(1 分)

8 证法 1. 归纳地定义 $A_0 = A, A_{n+1} = f(A_n) \neq \varnothing$, 则 $A \subset X \implies A_{n+1} \subset A_n$ 均为紧集, 且 X Hausdorff $\implies A_n$ 闭集. 考虑 $A = \bigcap_{n \in \mathbb{N}} A_n$, ············ (5 分)

则由推论 2.1.7, $A \neq \varnothing$. ············ (1 分)

下证 $f(A) = A$ 为所求.

一方面, $f(A) \subset \bigcap_{n \in \mathbb{N}} f(A_n) = \bigcap_{n \in \mathbb{N}} A_{n+1} = A$. ············ (2 分)

另一方面, $\forall y \in A, \forall n \in \mathbb{N}, y \in f_{n+1}(X) \iff f^{-1}(y) \cap f_n(X)$ 为非空闭集. 考虑 $f^{-1}(y) \supset f^{-1}(y) \cap A \supset \cdots \supset f^{-1}(y) \cap A_n \supset \cdots$. 则由推论 2.1.7, $f^{-1}(y) \cap A = \bigcap_{n \in \mathbb{N}} (f^{-1}(x) \cap A_n) \neq \varnothing \iff \exists x \in A$ s.t. $f(x) = y$, 即 $A \subset f(A)$. ············ (2 分)

证法 2. 考虑偏序集 $(\mathscr{J} = \left\{\varnothing \neq \text{ 闭子集 } A \subset X \colon f(A) \subset A\right\}, \prec)$, 其中 $A_1 \prec A_2 \iff A_1 \supset A_2$. ············ (5 分)

则由命题 2.1.6, $\forall$ 全序子集 $\{A_\alpha\}_\alpha, \bigcap_\alpha A_\alpha \neq \varnothing$, ············ (2 分)

且 $f\left(\bigcap_\alpha A_\alpha\right) \subset \bigcap_\alpha f(A_\alpha) \subset \bigcap_\alpha A_\alpha$, 即 $\bigcap_\alpha A_\alpha$ 为 $\{A_\alpha\}_\alpha$ 上界. ············ (1 分)

故由 Zorn 引理, $\mathscr{J}$ 有极大元 B. 若 $f(B) \subsetneq B$, 则 $\varnothing \neq f(f(B)) \subset f(B)$ 为 Hausdorff 空间的紧子集, 故闭 $\implies f(B) \in \mathscr{J}$, 且 $B \supset f(B)$, 与 B 极大性矛盾! 故 $B = f(B)$ 为所求. ············ (2 分)