第 5 章 拓扑学 (2024 秋季) 期末

练习 5.1(3 POINTS EACH, 30 POINTS IN TOTAL) Reading comprehension. Here is a part from our course notes.

> 下面证明Brower 区域不变性定理. 因为开集是一个局部条件, 且开集在平移和伸缩后仍是开集, 所以只要证明如下的"局部版本":
>
> **定理 5.1 (Brower 区域不变性定理, 局部版本)**
> 对于任意连续单射 $f\colon \overline{B^n} \to \mathbb{R}^n$, 点 $f(0)$ 位于 $f(\overline{B^n})$ 内部.
>
> **证明** 因为$\overline{B^n}$ 是紧的,$f(\overline{B^n})$ 是 Hausdorff 的,所以可逆连续映射 $f\colon \overline{B^n} \to f(\overline{B^n})$ 是一个 同胚 . 因为$f(\overline{B^n})$ 是紧集, 从而也是闭集,所以连续映射 $f^{-1}\colon f(\overline{B^n}) \to \overline{B^n}$ 可被扩张为连续映射 $g\colon \mathbb{R}^n \to \mathbb{R}^n$ 使得在 $f(\overline{B^n})$ 上有 $g = f^{-1}$.
>
> 设 $f(0)$ 不是 $f(\overline{B^n})$ 的 内点 , 则对于任意的 $\varepsilon > 0$, 存在 $c \in \mathbb{R}^n \backslash f(\overline{B^n})$ 使得 $|c - f(0)| < \varepsilon$. 记
>
> $$\Sigma_1 = \{y \in f(\overline{B^n})\colon |y - c| \geqslant \varepsilon\}, \quad \Sigma_2 = \{y \in \mathbb{R}^n\colon |y - c| = \varepsilon\}, \quad \Sigma = \Sigma_1 \bigcup \Sigma_2,$$
>
> 则对于任意 $y \in \Sigma$ 均有 $g(y) \neq 0$. 由于 g 连续而Σ_1 紧, 所以存在 $\delta > 0$ 使得对任意的 $y \in \Sigma$, 都有 $|g(y)| \geqslant \delta$.
>
> 另一方面, 由于Σ 是紧的 Hausdorff 空间 , 根据Stone-Weierstrass 定理, 存在"分量都是多项式的映射"……

1. $(3 \times 3')$Write down the definitions of 同胚 , 内点 and Hausdorff 空间 .
   - "A maps $f\colon X \to Y$ is a 同胚 between topological spaces" means:
   - "A point $p \in A$ is a 内点 of a subset A in a topological space X" means:
   - "A topological space X is a Hausdorff 空间 " means:
2. $(4 \times 1.5')$Here are some useful criteria for a set to be compact:
   (a). The image of a compact set under continuous map is compact;
   (b). Any finite union of compact set is compact;
   (c). Any closed subset of a compact set is compact;
   (d). Any bounded closed set in Euclidean space is compact;
   (e). Any product of compact spaces is compact.

   Explain why the four sets above (marked with underlines) are compact:
   - The reason for $\overline{B^n}$ 是紧的 is:
   - The reason for $f(\overline{B^n})$ 是紧的 is:
   - The reason for Σ_1 是紧的 is:
   - The reason for Σ 是紧的 is:
3. $(2 \times 3')$Write down the full statements of two theorems (marked with wavelines) above.
   - Brower 区域不变性定理:
   - Stone-Weierstrass 定理:
4. $(3 \times 3')$There are three 所以 …… above. Fill in the blanks.
   - The reason for 所以可逆连续映射 $f\colon \overline{B^n} \to f(\overline{B^n})$ 是一个同胚 is:

     A corollary of the closed mapping lemma: If X is ___, Y is ___, and $f\colon X \to Y$ is continuous and ___, then f is a homeomorphism.
   - The reason for 所以 $f^{-1}\colon f(\overline{B^n} \to \overline{B^n})$ 可被扩张为连续映射 $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is:

     Tietze extension theorem: If X is ___, $A \subseteq X$ is ___ and $f\colon A \to \mathbb{R}$ is ___, then f admits a continuous extension $g\colon X \to \mathbb{R}$.
   - The reason for 所以存在 $\delta > 0$ 使得对于任意的 $y \in \Sigma_0$ 都有 $|g(y)| \geqslant \delta$ is

     The extreme value theorem: Let $f\colon X \to \mathbb{R}$ be ___, $A \subseteq X$ is ___, then $f(A)$ is bounded in $\mathbb{R}$ and ___.

5. $(1 \times 3')$Write down a proof for the statement 则对于任意 $y \in \Sigma$, 均有 $g(y) \neq 0$.

✍ **练习 5.2(2 POINTS EACH, 20 POINTS IN TOTAL)** Which of the following statements are correct? Write a "T" before each correct statement, and write an "F" before each wrong statement. (**No details needed**)

1. Let X be a topological space, and $A \subseteq X$, then the subspace topology on A is the weakest topology on A so that the inclusion map $\iota\colon A \hookrightarrow X$ is continuous.
2. Let A, B are subsets of a topological space X. If A is homeomorphic to B, then $\overline{A}$ is homeomorphic to $\overline{B}$.
3. If X is Hausdorff and $Y = X/\sim$ is a quotient space of X, equipped with the quotient topology, then Y is Hausdorff.
4. Suppose a map $f\colon X \to Y$ satisfies "for any compact subset $K \subseteq X$, the restriction $f|_K\colon K \to Y$ is continuous", then $f\colon X \to Y$ is continuous.
5. Any closed subset in a paracompact(仿紧) space is paracompact.
6. If each X_α is connected, then $(\prod_\alpha X_\alpha, \mathscr{T}_{box})$ is connected.
7. A locally Euclidean space is connected if and only if it is path connected.
8. Any topological space can be embedded into a contractible(可缩) space.
9. The boundary circle of a Mbius banc is a retract(收缩核) of the Mbius band.
10. A topological space X is semi-locally simply connected(半局部单连通) if and only if for any $x \in X$, there is an open neighborhood U of x such that $\pi_1(U, x) = \{e\}$.
11. For any Jordan curve J in $\mathbb{S}^2$, $\mathbb{S}^2 \setminus J$ has exactly two connected components.

✍ **练习 5.3(4 POINTS EACH, 20 POINTS IN TOTAL)** Write down examples that satisfy the given requirements: (**No details needed**)

1. A surjective continuous map $f\colon \mathbb{Q} \to \mathbb{Z}$. both $\mathbb{Q}$ and $\mathbb{Z}$ are equipped with the standard topology
2. A countable family of subsets A_α in $\mathbb{R}^2$ such that $\bigcup_\alpha \overline{A_\alpha} \neq \overline{\bigcup_\alpha A_\alpha}$.
3. A topological space that is compact but not sequentially compact.
4. A topology $\mathscr{T}$ on $\mathbb{R}$ so that $(\mathbb{R}, \mathscr{T})$ is separable(可分的), first countable but not second countable.[You need to write down $\mathscr{T}$, or at least a basis of $\mathscr{T}$]
5. A covering map $f\colon (0, 2025) \to \mathbb{S}^1$.
6. A continuous injective map $f\colon \mathbb{R} \to \mathbb{R}^2$ such that $f\colon \mathbb{R} \to f(\mathbb{R})$ is not a homeomorphism.

✍ **练习 5.4(15 POINTS.)** Consider the following subsets in $\mathbb{R}^2$:

$$X = \{(x, y, z)\colon x^2 + y^2 + z^2 = 1, z > 0\},$$
$$X_n = X \bigcup \{(x, y, z)\colon x^2 + y^2 \leqslant 1, z = 0, 1, \cdots, n\},$$
$$X_\infty = X \bigcup \{(x, y, z)\colon x^2 + y^2 \leqslant 1, z = 0, 1, \cdots\}.$$

1. Write down the fundemental group of X and X_0, with brief explanation.
2. Compute the fundemental group of X_n for $n \in \mathbb{N}$ vie Van Kampen theorem.
3. What is the fundemental group of X_∞? Prove your assertion.

✍ **练习 5.5(15 POINTS)** Recall: two topological space X and Y are homotopy equivalent is there exists continuous maps $f\colon X \to Y$ and $g\colon Y \to X$, so that $g \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$.

1. Explain the meaning of the symbol $\sim$ (homotopy equivalence between maps).
2. Show that compactness is not invariant under homotopy equivalence by proving (WITH ALL DETAILS) that $\overline{B^n}$ is homotopy equivalant to $\mathbb{R}^n$.
3. Suppose X and Y are homotopy equivalent. Prove: If X is path connected, then Y is path connected.

✍ **练习 5.6(20 POINTS)** [Recall: two metrics on a set are topologically equivalent if they generate the same topology.]

1. Prove: the following two bounded metrics

$$d_1(x, y) = \min\{1, |x - y|\}, \quad d_2(x, y) = \left|\frac{x}{1+x} - \frac{y}{1+y}\right|$$

on $\mathbb{R}$, are topologically equivalent. No need to prove they are bounded metrics on $\mathbb{R}$.

2. For any set S and any bounded metric space (X, d), show that

$$D(f, g) := \sup_{x \in S} d(f(x), g(x))$$

is a metric on the set of maps $\mathcal{M}(S, X) = \{f\colon S \to X\colon f \text{ is a map}\}$.

3. Let $S = X = \mathbb{R}$ and consider the maps $f_n(x) = \begin{cases} x, & x < n; \\ n, & x \geqslant n \end{cases}$. Let $f_\infty(x) = x$ be the identity maps. Prove:$f_n \to f_\infty$ in respect to $D_2(f, g) = \sup_{x \in \mathbb{R}_+} d_2(f(s), g(s))$.

4. Let $D_1(f, g) = \sup_{x \in \mathbb{R}_+} d_1(f(x), g(x))$. Prove: $D_1(f_n, f_\infty) = 1$ for all n. Then write one sentence that summarize what you observed from this problem.

✍ **练习 5.7(10 POINTS)**

1. Prove: There exists no continuous injective map from $\mathbb{S}^2 \to \mathbb{T}^2$.
2. Prove: There exists no continuous injective map from $\mathbb{T}^2 \to \mathbb{S}^2$.

✍ **练习 5.8(15 POINTS)** Consider the three following subsets in $\mathbb{R}^2$:

$$X_1 = \{(x, \sin\frac{2\pi}{x}: 0 < x \leqslant 1)\},$$

$$X_2 = \{(0, y)\colon -1 \leqslant y \leqslant 1\},$$

$$X_3 = \{(x, y)\colon \left(x - \frac{1}{2}\right)^2 + \left(y + \frac{15}{16}\right)^2 = ()^2, y \leqslant 0\}.$$

The set $X := X_1 \cup X_2$ is known as "topologist's sine curve", and the set $Y = X_1 \cup X_2 \cup X_3$ is konwn as "Warsaw circle".

1. Construct a continuous map $f\colon X_1 \to X_1$ that has no fixed point.
2. Show that any continuous map $f\colon X \to X$ has a fixed point.
3. Show that Y is simply connected but not contractible.