2024 Spring Representation Theory of Groups and Associative Algebras Final Exam

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1. (20 points) Determining the character table of the following finite group.

$$\mathbb{Z}_3 \ltimes \mathbb{Z}_4 = \{a, b \mid a^3 = b^4 = aba^2b^3 = e\}$$

2. (40 points) The character table of group G is as follows. ρ_i is the character of irreducible representation V_i of G.

class	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
ρ_1	1	1	1	1	1	1	1	1	1	1
ρ_2	1	1	-1	1	1	1	1	-1	-1	-1
ρ_3	2	2	0	-1	2	2	2	0	0	0
ρ_4	3	3	-1	0	-1	-1	-1	-1	1	1
ρ_5	3	3	1	0	-1	-1	-1	1	-1	-1
ρ_6	3	-1	1	0	$-1+2i$	$-1-2i$	1	-1	$-i$	i
ρ_7	3	-1	-1	0	$-1+2i$	$-1-2i$	1	1	i	$-i$
ρ_8	3	-1	-1	0	$-1-2i$	$-1+2i$	1	1	$-i$	i
ρ_9	3	-1	1	0	$-1-2i$	$-1+2i$	1	-1	i	$-i$
ρ_{10}	6	-2	0	0	2	2	-2	0	0	0

(2.1) (10 points) Calculate $|C_i|$ for $i = 1, 2, \ldots, 10$.

(2.2) (10 points) Determine all normal subgroups.

(2.3) (10 points) Write down the non-trivial quotient group of G of the greatest order and its character table.

(2.4) (10 points) Is G Sovable? Is G nilpotent?

(2.5) (10 points) Write $V_5 \otimes V_{10}$ as a direct sum of irreducible representations.

3. (20 points) Suppose A is a semisimple algebra and $A = \prod_{i=1}^{s} A_i$ where $A_i (i = 1, 2 \ldots, s)$ are simple algebras.

(3.1) (10 points) Prove: $\dim Z(A) \geqslant s$

(3.2) (10 points) Prove: $\dim Z(A) = s$ if and only if $\mathbb{F}$ is a splitting field of A.

4. (20 points) Suppose G is the cyclic group of order 6, $\mathbb{F}_q$ is the finite field with q elements. Denote the gourp algebra of G over $\mathbb{F}_q$ as $A = \mathbb{F}_q[G]$.

(4.1) (10 points) Suppose $q = 2$, prove:

$$\frac{A}{\operatorname{rad} A} = \mathbb{F}_2 \oplus \mathbb{F}_4$$

(4.2) (10 points) Suppose $q = 3$, prove:

$$\frac{A}{\operatorname{rad} A} = \mathbb{F}_3 \oplus \mathbb{F}_3$$