# 2024 Spring Riemann Geometry Final

Date: 13 June 2024

# Problem 1. (3 points each, 30 points in total.)

Here is a classical inequality and its Riemannian geometry proof.

**Theorem 1** (Wirtinger inequality). .Let $f : [0, \pi] \to \mathbb{R}$ be a $C^2$ smooth function with $f(0) = f(\pi) = 0$. Then

$$\int_0^\pi (f(t))^2 \, dt \le \int_0^\pi (f'(t))^2 \, dt,$$

and equality holds if and only if $f(t) = c \sin t$ for some constant c.

*Proof.* Let $\gamma : [0, \pi] \to S^2$ be a normal geodesic on the round sphere $S^2$ from the north pole $p$ to the south pole $q$. Take a unit vector $v_0 \perp \dot\gamma(0)$ is $T_pS^2$, and parallel transport it to get a parallel vector field $v(t)$ along $gamma$, then $v(t) \perp \dot\gamma(t)$ for all $t$. Consider the vector field $X(t) = f(t)v(t)$ along $\gamma$, then $I(X, X) = \int_0^\pi ((f'(t))^2 - (f(t))^2) \, dt$. Since $q = \gamma(\pi)$ is the first conjugate point of $p = \gamma(0)$ along $gamma$, we have $I(X, X) \ge 0$. For the equality to hold, X must be a Jacobi field. Since $v(t) \perp \gamma(t)$, X has to be normal and thus $X(t) = c(\sin t)v(t)$. $\square$

Answer the following questions:

1. (6× 3') Write down the definitions of the following concepts that appeared above:

   geodesic: A smooth curve $\gamma : [a, b] \to M$ is called a geodesic if:

   parallel: A smooth vector field $X$ along $\gamma$ is called parallel if:

   Jacobi field: A smooth vector field $V$ along $\gamma$ is called a Jacobi field if:

   conjucate point: We say $q = \gamma(t_2)$ is conjucate to $p = \gamma(t_1)$ along $\gamma$ if:

   normal: A geodesic $\gamma$ is called a normal geodesic if:

   normal: A Jacobi field $V$ is called a normal Jacobi field if:

2. (3') In the proof there is a sentence "... then $v(t) \perp \dot\gamma(t)$ for all $t$".Why?
3. (3') Prove $I(X, X) = \int_0^\pi ((f'(t))^2 - (f(t))^2) \, dt$.
4. (3') In the last step we used the following result:

   if $X(t) = f(t)v(t)$ is a normal Jacobi field along a geodesic $\gamma : [0, t] \to S^2$, where $v$ is parallel along $\gamma$, then there exists constant $c$ and $d$ such that

   $$f(t) =$$

   .
5. (3') In the proof we write "For the equality to hold, $X$ must be a Jacobi field". Why?
6. (3') According to Wirtinger inequality, what can you say about the first Dirichlet eigenvalue $\lambda_1$ of the 1-dimensional Riemannian manifold $M = [0, \pi]$?

# Problem 2. (2 points each, 20 points in total.)

Which of the following statements are correct? Write a "T" before each correct statement, and write an "F" before each wrong statement.

- (     ) Any smooth manifold $M$ admits a linear connection.
- (     ) A linear connection $\nabla$ on a smooth manifold $M$ is flat if and only if near any point there is a local flat frame.
- (     ) A Riemannian manifold $(M, g)$ has positive sectional curvature if and only if it has positive curvature operator.

- ( ) A compact Riemannian manifold $(M, g)$ has constant sectional curvature 1 if and only if it is isometric to the round sphere $S^m$ or its quotient $\mathbb{RP}^m$ (with induced metric).
- ( ) In Riemannian normal coordinates neighborhood $U$, one has $(\nabla_{\partial_i}\partial_j(x)) = 0$ for any $x \in U$.
- ( ) On any Riemannian manifold, if $d(p, q) = l$, then there is a geodesicof length $l$ connecting $p$ and $q$.
- ( ) A smooth curve $\gamma$ connecting $p$ and $q$ minimize the energy functional $E$ in $\mathcal{C}_{pq}$ if and only if it minimize the length functional $L$ in $\mathcal{C}_{pq}$.
- ( ) If $(M, g)$ is a complete Riemannian manifold with sectional curvature $K \leq 0$, then any $p \in M$ has no conjucate point along any geodesic.
- ( ) If $p$ is a cut point of $q$, then $q$ is a cut point of $p$.
- ( ) Suppose $M$ is compact. If $M$ admits a Riemannian metric with $K < 0$, then $M$ cannot admit a Riemannian metric with $Ric \geq 0$.
- ( ) $S^2 \times S^2$ admits a metric with $K \geq \frac{1}{4}$ and diam $> \pi$.

# Problem 3. (3 points each, 15 points in total.)

For each of the following, write down an example.

1. A Riemannian manifold that is not complete, but any two points can be connected by a minimizing geodesic.
2. A smooth manifold that admits no constant sectional curvature metric.
3. A simply-connected non-compact manifold that admits no metric of $K < 0$.
4. A 4-dimensional compact manifold that admits a metric of $Ric > 0$ but admits no metric of $K > 0$.
5. A 5302-dimensional compact manifold that admits no metric of $K < 0$.
6. A compact connected smooth manifold of dimension 2024 that admits no Riemannian metric with $Ric > 0$.

# Problem 4. (15 Points.)

Let $(M, g)$ be a Riemannian manifold and $\nabla$ the Levi-Civita connection. For any $\omega \in \Omega^1(M)$, we let $X = \sharp\omega$ be the vector field obtained via musical isomorphism.

1. For any $Y \in \Gamma^\infty(TM)$, express $\omega(Y)$ in terms of $X, Y$ (and the metric $g$).
2. Recall that $\nabla X$ can be viewed as $(0, 2)$-tensor

   $$\nabla X : \Gamma^\infty(TM) \times \Gamma^\infty(TM), \nabla X(Y, Z) :\ = \langle \nabla_Y X, Z\rangle.$$

   Show that $\nabla X$ is symmetric if and only if $\mathrm{d}\omega = 0$.

   [You may need the formula $\mathrm{d}\omega(Y, Z) = Y(\omega(Z)) - Z(\omega(Y)) - \omega([Y, Z])$.]

# Problem 5. (15 Points.)

1. Let $\gamma\colon \mathbb{R} \to M$ be a smooth curve in $(M, g)$ parametrized by arc length. Suppose there exists an isometry $\phi\colon (M, g) \to (M, g)$ so that

   $$\{p \mid \phi(p) = p\} = \mathrm{Im}(\gamma).$$

   Prove: $\gamma$ is a normal geodesic in $M$.
2. Give a geodesic $\gamma\colon \mathbb{R} \to M$ on some complete Riemannian manifold for which there is no isometry $\phi$ such that $\{p \mid \phi(p) = p\} = \mathrm{Im}(\gamma)$.

# Problem 6. (25 Points.)

Consider the paraboloid

$$M \quad = \quad \{(x, y, z) \mid z = x^2 + y^2\} \subset \mathbb{R}^3,$$

endowed with the induced Riemannian metric $g$. Let $p = (0, 0, 0)$.

1. Write down the Riemannian metric tensor $g$ in terms of coordinates $x, y$.

2. find the distance getween $p$ and $q = (x_9, y_0, x_0^2 + y_0^2)$.
   [You may need $\int \sqrt{1 + 4x^2} = \frac{1}{4} \ln(|\sqrt{4x^2 + 1} + 2x|) + \frac{1}{2} x \sqrt{1 + 4x^2} + C$.]

3. Find all Christoffel symbols.

4. Find the sectional curvature $K$ at any point.

5. Answer the following problems with brief explanation.

   (a) Let $\gamma \colon [0, +\infty) \to M$ be the curve that is the intersection of $M$ with " a half with boundary $x = y = 0$",normalized by arc length, is $\gamma$ a geodesic?

   (b) Are those curves in (a) rays?

   (c) Does $p$ has any conjugate point?

   (d) Is there any line in $(M, g)$?

# Problem 7. (10 Points.)

Let $(M, g)$ be a complete Riemannian manifold with $K \geq 0$. Suppose $\gamma_1, \gamma_2 \colon [0, +\infty) \to M$ be two normal geodesics with $\gamma_1(0) = \gamma_2(0) = p$ and suppose $\gamma_1$ is a ray. Moreover, assume the angle $\alpha$ between $\dot{\gamma}_1(0)$ and $\dot{\gamma}_2(0)$ is less than $\frac{\pi}{2}$. Prove:

$$d(p, \gamma_2(t)) \geq t \cos(\alpha).$$

# Problem 8. (15 Points.)

1. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space of dimension $m$. Let $R \colon V \times V \to \mathbb{R}$ be a symmetric bilinear form, and $S^{m-1} = \{v \in V \mid ||v| = 1\}$ be the unit sphere in $V$,with area element $\mathrm{d}A$ and area $\omega_{m-1} \colon = \int_{S^{m-1}} \mathrm{d}A$. Prove

$$\frac{1}{\omega_{m-1}} \int_{S^{m-1}} R(v, v) \, \mathrm{d}A = \frac{1}{m} \mathrm{Tr}(R).$$

   [This problem is harder than others.]

2. Write down two consequences in Riemannian geometry.