2024 Spring Lie Algebras And Their Representations Midterm

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1. (20 points) Consider the following 3-dimensional Lie algebra $L = \mathbb{R}x \oplus \mathbb{R}y \oplus \mathbb{R}z$ defined by:

$$[x, y] = 2x + 2y, [y, z] = 2y + 2z, [z, x] = 2z + 2x$$

   (1.1) (10 points) Verify L is a Lie algebra.

   (1.2) (10 points) Is L isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ as Lie algebras ?

2. (15 points) Determine all ad-nilpotent elements in $\mathfrak{sl}_2(\mathbb{C})$.

3. (25 points) Let V be a L-module. Let $\omega : V \times V \to \mathbb{C}$ be a non-degenerate bilinear form. Define a vector space homomorphism as follows.

$$T_\omega : V \to V^* : v \to \omega(v, \cdot)$$

   (3.1) Prove: T_ω is isomorphic.

   (3.2) Prove: V is an irreducible L-module implies V^* is an irreducible L-module.

   (3.3) Suppose $\omega(l.x, y) + \omega(x, l.y) = 0$ for any $l \in L$ and $x, y \in V$. Prove that T_ω is a L-module isomorphism.

4. (40 points) Suppose $\mathfrak{g}$ is a finite dimensional Lie algebra over $\mathbb{C}$. View $\mathfrak{g}$ as a $\mathbb{C}$-vector space and equip $\mathbb{C}$-vector space $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[x]$ with a bracket operation:

$$[g_1 \otimes_{\mathbb{C}} f_1(x), g_2 \otimes_{\mathbb{C}} f_2(x)] = [g_1, g_2] \otimes_{\mathbb{C}} f_1(x)f_2(x)$$

   (4.1) (10 points) Verify $L(\mathfrak{g}) = (\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[x], [\cdot, \cdot])$ is a Lie algebra.

   (4.2) (10 points) Prove: $\phi_a : L(\mathfrak{g}) \to \mathfrak{g} : g \otimes_{\mathbb{C}} f(x) \mapsto f(a)g$ is a Lie algebra homomorphism.

   (4.3) (10 points) Give a concrete automorphism of $L(\mathfrak{sl}_2(\mathbb{C}))$ that isn't an inner automorphism.

   (4.4) (10 points) Determine all Lie algebra homomorphisms $L(\mathfrak{sl}_2(\mathbb{C})) \to \mathfrak{sl}_2(\mathbb{C})$.

   (4.5) Determine all irreducible $L(\mathfrak{sl}_2(\mathbb{C}))$-modules.

Some solutions:

1(2): equitable basis of $\mathfrak{sl}_2(\mathbb{F})$, `https://arxiv.org/pdf/0810.2066`.

4(5): For a simple finite dimensional Lie algebra over $\mathbb{C}$ denoted as $\mathfrak{g}$, we have the following bijection:

$$\{\text{simple } L(\mathfrak{g})\text{-module } V\} \longleftrightarrow \{\text{a simple } \mathfrak{g}\text{-module } V \text{ and a scarlar endomorphism } \lambda_a : V \to V : v \mapsto av\}$$

`https://math.stackexchange.com/questions/4808441/simple-modules-of-mathcalg-otimes-mathbbcx-for-a-lie-al`