2024 Spring Lie Algebras And Their Representations Final

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1. A Cartan subalgebra (abbreviated as CSA) of a Lie algebra L is a nilpotent subalgebra that equals its normalizer in L.
   (1.1) Prove CSA of a Lie algebra L is maximal nilpotent i.e. not properly included in any nilpotent subalgebra of L.
   (1.2) Show that the converse of (1.1) is false: give an example of a maximal nilpotent subalgebra of semisimple Lie algebra L that isn't a CSA.
   (1.3) For semisimple Lie algebra L, suppose H is a maximal toral subalgebra of L and Φ is the set of roots relative to H.
   Prove: for $h \in L$, $C_L(h) = H$ if and only if $h \in H$ and $\alpha(h) \neq 0$ for any $\alpha \in \Phi$.

2. Let Φ be a root system and $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be an ordered base of Φ. The Cartan matrix of Δ is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

   (2.1) Draw the Dynkin diagram.
   (2.2) Write down all positive roots.
   (2.3) Write down the fundamental dominant weights in terms of $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.
   (2.4) Write down the fundamental group Λ/Λ_r.

3. Let H be a maximal toral subalgebra of semisimple Lie algebra L and Let Φ be the set of all roots.
   Prove that $|\Phi|$ is even and $2\dim H \leqslant |\Phi|$. When do we have $2\dim H = |\Phi|$?

4. Let Φ be an irreducible root system of Euclidean space E. Denote the Weyl group of it as $\mathcal{W}$.
   (4.1) Suppose $S \subseteq \Phi$ and $\Phi \subseteq \mathbb{R}_{\geqslant 0}S \cup \mathbb{R}_{\leqslant 0}S$. Is S a base of Φ?
   (4.2) Suppose there is a root system isomorphism $f : (\Phi, E) \to (\Phi', E')$.
   Prove: for any $\sigma \in \mathcal{W}$,
$$\frac{(\sigma(\alpha), \sigma(\alpha))}{(\alpha, \alpha)} = \frac{(f(\sigma(\alpha)), f(\sigma(\alpha)))}{(f(\alpha), f(\alpha))}$$

5. Suppose we have a decomposition of root system $\Phi = P \cup -P$ for some $P \subseteq \Phi$. For any $\alpha, \beta \in P$, $\alpha + \beta \in \Phi$ implies $\alpha + \beta \in P$.
   Prove that there exists a base Δ of Φ such that $\Phi^+ \subseteq P$.