Harmonic Analysis Final Exam

2024 Spring

Each problem is worth 25 points.

Problem 1: Let $u \in L^2(\mathbb{R})$. We set

$$U(x,\xi) = \int e^{-(x+i\xi-y)^2/2} u(y) dy$$

Show that $U(x,\xi)$ is well-defined on $\mathbb{R}^2$ and there exists a positive constant C such that

$$\iint |U(x,\xi)|^2 e^{-\xi^2} dx d\xi = C \int |u(y)|^2 dy$$

for all $u \in L^2(\mathbb{R})$.

Problem 2(GTM249.P384, Exercise 5.4.4): (a) Prove that for all $x, y \in \mathbb{R}^n$ that satisfy $0 \neq x \neq y$, we have

$$\left| \frac{x-y}{|x-y|} - \frac{x}{|x|} \right| \leq 2 \frac{|y|}{|x|}$$

(b) Let Ω be an integrable function with mean value zero on the sphere $\mathbb{S}^{n-1}$. Suppose that Ω satisfies a Hölder condition of order $0 < \alpha < 1$ on $\mathbb{S}^{n-1}$. This means that

$$|\Omega(\theta_1) - \Omega(\theta_2)| \leq B_0 |\theta_1 - \theta_2|^\alpha$$

for all $\theta_1, \theta_2 \in \mathbb{S}^{n-1}$. Prove that $K(x) = \Omega(x/|x|)/|x|^n$ satisfies

$$\int_{|x| \geq 2|y|} |K(x-y) - K(x)| \, dx < C$$

Problem 3(GTM249.P89, Example 2.1.8): Let $R > 0$ and $\chi_{B(0,R)}$ be the characteristic function of the ball $B(0,R)$. Prove that

$$\frac{R^n}{(|x|+R)^n} \leq M\left(\chi_{B(0,R)}(x)\right) \leq \frac{6^n R^n}{(|x|+R)^n}$$

Problem 4(GTM249.P170, Exercise 2.6.7 without hint): Show that there is a constant $C < \infty$ such that for all non-integers $\gamma > 1$ and all $\lambda, b > 1$, we have

$$\left| \int_0^b e^{i\lambda t^\gamma} dt \right| \leq \frac{C}{\lambda^\gamma}$$