一、设曲面 $S: \vec{r}=\vec{r}(u,v)$ 第一基本形式为 $\mathrm{I}=E\,du\,du+2F\,du\,dv+G\,dv\,dv$，求 Christoffel 符号 $\Gamma_{12}^2$.

解：记 $(u^1,u^2)=(u,v)$，则

$$\Gamma_{12}^2=\frac{1}{2}g^{21}\left(\frac{\partial g_{11}}{\partial u^2}+\frac{\partial g_{12}}{\partial u^1}-\frac{\partial g_{12}}{\partial u^1}\right)+\frac{1}{2}g^{22}\left(\frac{\partial g_{21}}{\partial u^2}+\frac{\partial g_{22}}{\partial u^1}-\frac{\partial g_{12}}{\partial u^2}\right)$$

$$=\frac{1}{2}g^{21}\frac{\partial g_{11}}{\partial u^2}+\frac{1}{2}g^{22}\frac{\partial g_{22}}{\partial u^1}$$

现 $(g_{ij})=\begin{pmatrix}E & F\\ F & G\end{pmatrix}$，$(g^{ij})=\begin{pmatrix}E & F\\ F & G\end{pmatrix}^{-1}=\dfrac{1}{EG-F^2}\begin{pmatrix}G & -F\\ -F & E\end{pmatrix}$

故 $$\Gamma_{12}^2=\frac{1}{2}\left(\frac{-F}{EG-F^2}\right)\frac{\partial E}{\partial v}+\frac{1}{2}\left(\frac{E}{EG-F^2}\right)\frac{\partial G}{\partial u}$$

$$=\frac{1}{2}\frac{1}{EG-F^2}\left(E\frac{\partial G}{\partial u}-F\frac{\partial E}{\partial v}\right).$$

二、设 $\{r;e_1,e_2,e_3\}$ 为曲面 $S:\vec{r}=\vec{r}(u,v)$ 的一个正交标架，其中 $e_3$ 是曲面单位法向量场。设 $\overline{e_1}=\cos\theta\, e_1+\sin\theta\, e_2$，$\overline{e_2}=-\sin\theta\, e_1+\cos\theta\, e_2$，$\overline{e_3}=e_3$，其中 $\theta=\theta(u,v)$ 是任意可微函数。记 $\omega^i=\langle d\vec{r},e_i\rangle$，$\overline{\omega}^i=\langle d\vec{r},\overline{e_i}\rangle$，$\omega_i^j=\langle de_i,e_j\rangle$，$\overline{\omega}_i^j=\langle d\overline{e}_i,\overline{e}_j\rangle$.

求 $\overline{\omega}_1^2-\omega_1^2$ 与 $(\overline{\omega}^1\overline{\omega}_2^3-\overline{\omega}^2\overline{\omega}_1^3)-(\omega^1\omega_2^3-\omega^2\omega_1^3)$.

解：已有 $\begin{pmatrix}\overline{e_1}\\ \overline{e_2}\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix}\triangleq A\begin{pmatrix}e_1\\ e_2\end{pmatrix}$，

因 $$d\vec{r}=(\omega^1\ \ \omega^2)\begin{pmatrix}e_1\\ e_2\end{pmatrix}$$

$$=(\overline{\omega}^1\ \ \overline{\omega}^2)\begin{pmatrix}\overline{e}_1\\ \overline{e}_2\end{pmatrix}=(\overline{\omega}^1\ \ \overline{\omega}^2)A\begin{pmatrix}e_1\\ e_2\end{pmatrix}$$

有 $(\omega^1\ \omega^2)=(\overline{\omega}^1\ \overline{\omega}^2)A$，或者 $(\overline{\omega}^1\ \overline{\omega}^2)=(\omega^1\ \omega^2)A^{-1}$.

$d\bar{e}_1 = -\sin\theta\, d\theta\, e_1 + \cos\theta\, de_1 + \cos\theta\, d\theta\, e_2 + \sin\theta\, de_2$

$d\bar{e}_2 = -\cos\theta\, d\theta\, e_1 - \sin\theta\, de_1 - \sin\theta\, d\theta\, e_2 + \cos\theta\, de_2$

则 $\bar{\omega}_1^2 = \langle d\bar{e}_1, \bar{e}_2 \rangle = \langle -\sin\theta\, d\theta\, e_1 + \cos\theta\, de_1 + \cos\theta\, d\theta\, e_2 + \sin\theta\, de_2,$

$-\sin\theta\, e_1 + \cos\theta\, e_2 \rangle$

$= (\sin^2\theta\, d\theta + \cos^2\theta\, d\theta) + \cos^2\theta\, \omega_1^2 - \sin^2\theta\, \omega_2^1$

$= d\theta + \omega_1^2$

故 $\bar{\omega}_1^2 - \omega_1^2 = d\theta$.

$\bar{\omega}_1^3 = \langle d\bar{e}_1, \bar{e}_3 \rangle = \langle d\bar{e}_1, e_3 \rangle = \cos\theta\, \omega_1^3 + \sin\theta\, \omega_2^3$

$\bar{\omega}_2^3 = \langle d\bar{e}_2, \bar{e}_3 \rangle = \langle d\bar{e}_2, e_3 \rangle = -\sin\theta\, \omega_1^3 + \cos\theta\, \omega_2^3$

从而: $\begin{pmatrix} \bar{\omega}_2^3 \\ -\bar{\omega}_1^3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \omega_2^3 \\ -\omega_1^3 \end{pmatrix} = A \begin{pmatrix} \omega_2^3 \\ -\omega_1^3 \end{pmatrix}$,

故 $\bar{\omega}^1 \bar{\omega}_2^3 - \bar{\omega}^2 \bar{\omega}_1^3 = (\bar{\omega}^1 \ \ \bar{\omega}^2) \begin{pmatrix} \bar{\omega}_2^3 \\ -\bar{\omega}_1^3 \end{pmatrix}$

$= (\omega^1 \ \ \omega^2) A^{-1} A \begin{pmatrix} \omega_2^3 \\ -\omega_1^3 \end{pmatrix}$

$= (\omega^1 \ \ \omega^2) \begin{pmatrix} \omega_2^3 \\ -\omega_1^3 \end{pmatrix} = \omega^1 \omega_2^3 - \omega^2 \omega_1^3$.

故 $(\bar{\omega}^1 \bar{\omega}_2^3 - \bar{\omega}^2 \bar{\omega}_1^3) - (\omega^1 \omega_2^3 - \omega^2 \omega_1^3) = 0$.

综上: 1°: $\bar{\omega}_1^2 - \omega_1^2 = d\theta$

2°: $(\bar{\omega}^1 \bar{\omega}_2^3 - \bar{\omega}^2 \bar{\omega}_1^3) - (\omega^1 \omega_2^3 - \omega^2 \omega_1^3) = 0$.

三、是否存在曲面分别以如下$\phi_1$ $\phi_2$为第一、第二基本形式？说明理由。

(1) $\phi_1 = dudu + dvdv$ ，$\phi_2 = dudu + 3dvdv$

(2) $\phi_1 = dudu + dvdv$ ，$\phi_2 = -dudu$

(3) $\phi_1 = 4\cos^2 v\, dudu + dvdv$ ，$\phi_2 = dudu + 4\cos^2 v\, dvdv$ $(\cos v > 0)$

解：由曲面论基本定理，只须验证$\phi_1$ $\phi_2$是否满足Gauss-Codazzi方程：

· Gauss方程：$-\frac{1}{\sqrt{EG}}\left(\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right) = \frac{LN}{EG}$

· Codazzi方程：$L_v = HE_v$ 其中 $H = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right)$.

$N_u = HG_u$

这里 $\mathrm{I} = E\,dudu + G\,dvdv$，$\mathrm{II} = L\,dudu + N\,dvdv$.

(1) 令 $E=1$，$G=1$，$L=1$，$N=3$， Gauss方程左边 $=0$，右边 $=3$，不成立，

故不存在曲面

(2) 令 $E=1$，$G=1$，$L=-1$，$N=0$，则 ① Gauss方程：左边 $=0$，右边 $=0$ 成立

② Codazzi方程：左边 $=0$，右边 $=0$ 成立

故存在曲面

(3) 令 $E = 4\cos^2 v$，$G=1$，

$L = 1$ ，$N = 4\cos^2 v$，

$H = \frac{1}{2}\left(\frac{1}{4\cos^2 v} + 4\cos^2 v\right)$，

Gauss方程：左边 $= -\frac{1}{2\cos v}\left((2\cos v)_{vv}\right) = -\frac{-2\cos v}{2\cos v} = 1$，

右边 $= 1$ ，成立.

Codazzi方程； $L_v = 0$，但 $HE_v = \frac{1}{2}\left(\frac{1}{4\cos^2 v} + 4\cos^2 v\right)(-8\cos v\sin v) \neq 0$

故Codazzi方程不成立，不存在曲面.

四、设曲面S的主曲率$k_1, k_2$为光滑函数，曲面参数$(u,v)$使得$\vec{r}_u, \vec{r}_v$分别为主曲率$k_1, k_2$对应的主方向。设曲面一点$P_0$处，$k_1(P_0) > k_2(P_0)$，且$P_0$为$k_1$的局部极大值点，为$k_2$的局部极小值点。

(1) 记曲面S的第一、第二基本形式分别为$\mathrm{I} = E\,du\,du + G\,dv\,dv$，

$\mathrm{II} = L\,du\,du + N\,dv\,dv$，求解$k_1, k_2$表达式以及$\frac{\partial E}{\partial v}(P_0)$，$\frac{\partial G}{\partial u}(P_0)$的数值。

(2) 证明：$P_0$处的Gauss曲率$K(P_0) \le 0$。

(1) 解：一方面 $W\begin{pmatrix}\vec{r}_u\\ \vec{r}_v\end{pmatrix} = \begin{pmatrix}k_1 & \\ & k_2\end{pmatrix}\begin{pmatrix}\vec{r}_u\\ \vec{r}_v\end{pmatrix}$，另一方面

$$W\begin{pmatrix}\vec{r}_u\\ \vec{r}_v\end{pmatrix} = \begin{pmatrix}L & 0\\ 0 & N\end{pmatrix}\begin{pmatrix}E & 0\\ 0 & G\end{pmatrix}^{-1}\begin{pmatrix}\vec{r}_u\\ \vec{r}_v\end{pmatrix} = \begin{pmatrix}\frac{L}{E} & 0\\ 0 & \frac{N}{G}\end{pmatrix}\begin{pmatrix}\vec{r}_u\\ \vec{r}_v\end{pmatrix},$$

故 $k_1 = \frac{L}{E}$，$k_2 = \frac{N}{G}$。

因$P_0$是$k_1$局部极大值点，则 $0 = \left(\frac{L}{E}\right)_v(P_0) = \frac{L_vE - LE_v}{E^2}(P_0) = 0$，

故 $L_v(P_0) = \frac{L}{E}(P_0)E_v(P_0) = k_1(P_0)E_v(P_0)$ —(1)

由Codazzi方程 $L_v(P_0) = \frac{1}{2}(k_1(P_0) + k_2(P_0))E_v(P_0)$ —(2).

结合(1),(2)有：$\frac{1}{2}(k_1(P_0) - k_2(P_0))E_v(P_0) = 0 \xRightarrow[k_1(P_0) > k_2(P_0)]{} E_v(P_0) = 0$.

因$P_0$是$k_2$的局部极小值点，则 $0 = \left(\frac{N}{G}\right)_u(P_0) = \frac{N_uG - NG_u}{G^2}(P_0) = 0$,

故 $N_u(P_0) = \frac{N}{G}(P_0)G_u(P_0) = k_2(P_0)G_u(P_0)$ —(3)

由Codazzi方程：$N_u(P_0) = \frac{1}{2}(k_1(P_0) + k_2(P_0))G_u(P_0)$ —(4)

结合(3),(4)有：$\frac{1}{2}(k_2(P_0) - k_1(P_0))G_u(P_0) = 0 \xRightarrow[k_1(P_0) > k_2(P_0)]{} G_u(P_0) = 0$.

(2) 证明：因$P_0$是$k_1$局部极大值点，有：

$$0 \geqslant \left(\frac{L}{E}\right)_{vv}(P_0) = \left(\frac{L_vE - LE_v}{E^2}\right)_v(P_0) \xlongequal{E_v(P_0)=0} \frac{L_{vv}E - LE_{vv}}{E^2}(P_0) \text{ —(5)}$$

由Codazzi方程 $L_v = HE_v$ 关于$v$求导 $L_{vv}(P_0) = HE_{vv}(P_0)$ —(6)

结合(5),(6)知：$(HE - L)E_{vv}(P_0) \le 0$，

因：$(HE-L)(p_0) = E(p_0)\left(H(p_0)-\frac{L}{E}(p_0)\right) = E(p_0)\frac{1}{2}\left(k_2(p_0)-k_1(p_0)\right) < 0$

有 $E_{vv}(p_0) \geq 0$.

因 $p_0$ 是 $k_2$ 局部极小值点，有：

$$0 \leq \left(\frac{N}{G}\right)_{uu}(p_0) = \left(\frac{N_uG-NG_u}{G^2}\right)_u(p_0) \xlongequal{G_u(p_0)=0} \frac{N_{uu}G-NG_{uu}}{G^2}(p_0) \quad -(7)$$

又由 Codazzi 方程：$N_u = HG_u$ 关于 $u$ 求导：$N_{uu}(p_0) = HG_{uu}(p_0)$, $-(8)$

由 (7)、(8) 可知：$(HG-N)G_{uu}(p_0) \geq 0$.

因 $(HG-N)(p_0) = G(p_0)\left(H(p_0)-\frac{N}{G}(p_0)\right) = G(p_0)\frac{1}{2}\left(k_1(p_0)-k_2(p_0)\right) > 0$,

有 $G_{uu}(p_0) \geq 0$.

$$\text{故 } K(p_0) = -\frac{1}{\sqrt{EG}}\left(\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right)(p_0)$$

$$= -\frac{1}{\sqrt{EG}}\left(\left(\frac{\frac{1}{2}E_v}{\sqrt{EG}}\right)_v + \left(\frac{\frac{1}{2}G_u}{\sqrt{EG}}\right)_u\right)(p_0)$$

$$\xlongequal[G_u(p_0)=0]{E_v(p_0)=0} -\frac{1}{\sqrt{EG}}\left(\frac{1}{2}\frac{E_{vv}}{\sqrt{EG}} + \frac{1}{2}\frac{G_{uu}}{\sqrt{EG}}\right)(p_0)$$

$$= -\frac{1}{2}\frac{1}{EG}(E_{vv}+G_{uu})(p_0) \leq 0.$$

已证 $E_{vv}(p_0) \geq 0$ 且 $E(p_0) > 0$

$G_{uu}(p_0) \geq 0$ 　 $G(p_0) > 0$

五：判定曲面 $\vec{r}(u,v) = (u\cos v, u\sin v, \log u)$ 与

$\tilde{r}(x,y) = (x\cos y, x\sin y, y)$ 是否等距，并证明之.

解：曲面 $\vec{r}(u,v)$：$\vec{r}_u = (\cos v, \sin v, \frac{1}{u})$，$\vec{r}_v = (-u\sin v, u\cos v, 0)$.

$$E = \vec{r}_u\cdot\vec{r}_u = 1+\frac{1}{u^2},\quad F = \vec{r}_u\cdot\vec{r}_v = 0,\quad G = \vec{r}_v\cdot\vec{r}_v = u^2$$

$$\mathrm{I} = \left(1+\frac{1}{u^2}\right)du^2 + u^2dv^2,$$

Gauss 曲率 $K=-\frac{1}{\sqrt{EG}}\left(\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v+\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right)$

$$=-\frac{1}{\sqrt{1+u^2}}\left(\frac{u}{\sqrt{1+u^2}}\right)_u=-\frac{1}{(1+u^2)^2}$$

曲面 $\tilde{r}(x,y)$：$\tilde{r}_x=(\cos y,\sin y,0)$，$\tilde{r}_y=(-x\sin y,x\cos y,1)$，

$\tilde{E}=\tilde{r}_x\cdot\tilde{r}_x=1$，$\tilde{F}=\tilde{r}_x\cdot\tilde{r}_y=0$，$\tilde{G}=\tilde{r}_y\cdot\tilde{r}_y=x^2+1$

则 $\tilde{\mathrm{I}}=dx^2+(x^2+1)dy^2$，

Gauss 曲率 $\tilde{K}=-\frac{1}{\sqrt{\tilde{E}\tilde{G}}}\left(\left(\frac{(\sqrt{\tilde{E}})_y}{\sqrt{\tilde{G}}}\right)_y+\left(\frac{(\sqrt{\tilde{G}})_x}{\sqrt{\tilde{E}}}\right)_x\right)$

$$=-\frac{1}{\sqrt{1+x^2}}\left(\sqrt{x^2+1}\right)_{xx}=-\frac{1}{(1+x^2)^2}.$$

若 $r(u,v)$ 和 $\tilde{r}(x,y)$ 等距，由 Gauss 绝妙定理有 $K\underset{\text{参数对应}}{=}\tilde{K}$，故有

参数对应：$u=x$（假设 $x>0$），但

$$\mathrm{I}(u,v)=\left(1+\frac{1}{u^2}\right)du^2+u^2dv^2\ \underset{v=f(y)}{\overset{u=x}{=\!=}}\ \left(1+\frac{1}{x^2}\right)dx^2+x^2f'(y)^2dy^2$$

$$\neq dx^2+(1+x^2)dy^2=\tilde{\mathrm{I}}(x,y)$$

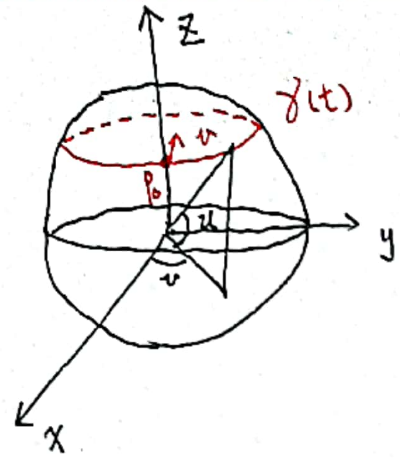
矛盾．故曲面 $\vec{r}(u,v)$ 和 $\tilde{r}(x,y)$ 不等距

六．在单位球面 $S^2$，求在 $P_0=\frac{\sqrt{2}}{2}(1,0,1)$ 处的切向量 $v=\frac{\sqrt{2}}{2}(-1,0,1)$ 沿曲线 $\gamma(t)=\frac{\sqrt{2}}{2}(\cos t,\sin t,1)$（$t\in[0,2\pi]$）平行移动回到 $P_0$ 处的切向量 $v$．

解：设球面 $S^2$ 参数方程：$\vec{r}(u,v)=(\cos u\cos v,\cos u\sin v,\sin u)$

$u\in(-\frac{\pi}{2},\frac{\pi}{2})$，$v\in(0,2\pi)$．

令 $u=\frac{\pi}{4}$，$v=t$ 即有 $S^2$ 上曲线 $\gamma(t)=\frac{\sqrt{2}}{2}(\cos t,\sin t,1)$，$P_0=\gamma(0)$．




$$\vec{r}_u = (-\sin u\cos v,\ -\sin u\sin v,\ \cos u)$$

$$\vec{r}_v = (-\cos u\sin v,\ \cos u\cos v,\ 0)$$

取正交标架 $e_1 = (-\sin u\cos v,\ -\sin u\sin v,\ \cos u)$

$$e_2 = (-\sin v,\ \cos v,\ 0)$$

有 $\omega_1 = du,\quad \omega_2 = \cos u\, dv,$

因 $\begin{cases} d\omega_1 = d(du) = 0 \\ d\omega_1 = \omega_{12}\wedge\omega_2 \end{cases}$ $\begin{cases} d\omega_2 = d(\cos u\, dv) = -\sin u\, du\wedge dv \\ d\omega_2 = \omega_{21}\wedge\omega_1 = \omega_1\wedge\omega_{12} \\ \qquad = du\wedge\omega_{12} \end{cases}$

故 $\omega_{12} = -\sin u\, dv$

限制在曲线 $\gamma(t)$ 上有：$e_1 = \frac{\sqrt{2}}{2}(-\cos t, -\sin t, 1)$ ，$\omega_{12} = -\frac{\sqrt{2}}{2}dt,$

$$e_2 = (-\sin t,\ \cos t,\ 0)$$

设 $V(t) = f^1(t)e_1 + f^2(t)e_2$ 是沿 $\gamma(t)$ 的 $S^2$ 上的向量场，沿 $\gamma(t)$ 平行移动，满足 $V(0) = v = \frac{\sqrt{2}}{2}(-1,0,1) = e_1$，即 $f^1(0)=1,\ f^2(0)=0$.

故 $0 = \frac{DV(t)}{dt} = \frac{df^1(t)}{dt}e_1 + f^1(t)\frac{De_1}{dt} + \frac{df^2(t)}{dt}e_2 + f^2(t)\frac{De_2}{dt}$

$$= \frac{df^1(t)}{dt}e_1 + f^1(t)\frac{\omega_{12}}{dt}e_2 + \frac{df^2(t)}{dt}e_2 + f^2(t)\frac{\omega_{21}}{dt}e_1$$

$$= \left(\frac{df^1(t)}{dt} + \frac{\sqrt{2}}{2}f^2(t)\right)e_1 + \left(\frac{df^2(t)}{dt} - \frac{\sqrt{2}}{2}f^1(t)\right)e_2$$

$\Longleftrightarrow$ 常微分方程组 $\begin{cases} \frac{df^1(t)}{dt} + \frac{\sqrt{2}}{2}f^2(t) = 0 \\ \frac{df^2(t)}{dt} - \frac{\sqrt{2}}{2}f^1(t) = 0 \\ f^1(0)=1,\ f^2(0)=0. \end{cases}$ 解得 $f^1(t) = \cos\left(\frac{\sqrt{2}}{2}t\right)$

$f^2(t) = \sin\left(\frac{\sqrt{2}}{2}t\right)$

故 $V(t) = \cos\left(\frac{\sqrt{2}}{2}t\right)e_1 + \sin\left(\frac{\sqrt{2}}{2}t\right)e_2$，则

$$v' = V(2\pi) = \cos(\sqrt{2}\pi)\frac{\sqrt{2}}{2}(-1,0,1) + \sin(\sqrt{2}\pi)(0,1,0)$$

$$= \left(-\frac{\sqrt{2}}{2}\cos(\sqrt{2}\pi),\ \sin(\sqrt{2}\pi),\ \frac{\sqrt{2}}{2}\cos(\sqrt{2}\pi)\right).$$

另法：设 $\beta(t)$ 是和 $e_1$ 的夹角，则可设 $V(t) = \cos\beta(t)\, e_1 + \sin\beta(t)\, e_2$,
$(V(0) = v = e_1)$

由 $V(t)$ 沿 $\gamma(t)$ 平行移动有：

$$0 = \frac{DV(t)}{dt} = -\sin\beta(t)\frac{d\beta}{dt} e_1 + \cos\beta(t)\frac{\omega_{12}}{dt} e_2$$
$$+ \cos\beta(t)\frac{d\beta}{dt} e_2 + \sin\beta(t)\frac{\omega_{21}}{dt} e_1$$
$$= -\sin\beta(t)\left(\frac{d\beta}{dt} + \frac{\omega_{12}}{dt}\right) e_1 + \cos\beta(t)\left(\frac{d\beta}{dt} + \frac{\omega_{12}}{dt}\right) e_2$$

则 $\frac{d\beta}{dt} + \frac{\omega_{12}}{dt} = \left\langle \frac{DV(t)}{dt}, -\sin\beta(t)\, e_1 + \cos\beta(t)\, e_2 \right\rangle = 0$,

故 $d\beta = -\omega_{12} = \frac{\sqrt{2}}{2} dt \Rightarrow \beta(2\pi) = \int_0^{2\pi} \frac{\sqrt{2}}{2} dt + \underbrace{\beta(0)}_{=0}$
$$= \sqrt{2}\pi$$

故 $v' = \cos(\sqrt{2}\pi)\, e_1(2\pi) + \sin(\sqrt{2}\pi)\, e_2(2\pi)$
$$= \left(-\frac{\sqrt{2}}{2}\cos(\sqrt{2}\pi),\ \sin(\sqrt{2}\pi),\ \frac{\sqrt{2}}{2}\cos(\sqrt{2}\pi)\right).$$

七. 求曲面 $\vec{r}(u,v)=(u\cos v,\ u\sin v,\ v)$ 上的测地线.

解: $\vec{r}_u=(\cos v,\ \sin v,\ 0)$ $\qquad E=\vec{r}_u\cdot\vec{r}_u=1,\quad F=\vec{r}_u\cdot\vec{r}_v=0$

$\vec{r}_v=(-u\sin v,\ u\cos v,\ 1)$ $\qquad G=\vec{r}_v\cdot\vec{r}_v=u^2+1$

设 $\vec{r}(s)=\vec{r}(u(s),v(s))$ 是 $\vec{r}(u,v)$ 中的测地线, $s$ 是弧长参数, $\theta$ 是切向量 $\frac{d\vec{r}(s)}{ds}$ 和 $e_1=\vec{r}_u$ 的夹角, 则由 Liouville 公式:

$$\frac{d\theta}{ds}=\frac{1}{2}\frac{1}{\sqrt{G}}\frac{\partial\ln E}{\partial v}\cos\theta-\frac{1}{2\sqrt{E}}\frac{\partial\ln G}{\partial u}\sin\theta=-\frac{1}{2}\frac{2u}{u^2+1}\sin\theta=-\frac{u}{u^2+1}\sin\theta \quad ①$$

$$\begin{cases}\dfrac{du}{ds}=\dfrac{1}{\sqrt{E}}\cos\theta=\cos\theta & ②\\[2ex] \dfrac{dv}{ds}=\dfrac{1}{\sqrt{G}}\sin\theta=\dfrac{1}{\sqrt{u^2+1}}\sin\theta & ③\end{cases}$$

$$\frac{①}{②}\Rightarrow\frac{d\theta}{du}=-\frac{u}{u^2+1}\frac{\sin\theta}{\cos\theta}\Rightarrow\frac{\cos\theta}{\sin\theta}d\theta=-\frac{u}{u^2+1}du$$

$$\Rightarrow\int\frac{d(\sin\theta)}{\sin\theta}=-\frac{1}{2}\int\frac{d(u^2+1)}{u^2+1}\Rightarrow\ln|\sin\theta|+\frac{1}{2}\ln(u^2+1)=\tilde{C},$$

从而: $\sin\theta\cdot\sqrt{u^2+1}=C$ (常数).

则 $\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-\left(\dfrac{C}{\sqrt{u^2+1}}\right)^2}=\dfrac{\sqrt{u^2+1-C^2}}{\sqrt{u^2+1}}$

故 $\dfrac{②}{③}\Rightarrow\dfrac{du}{dv}=\sqrt{u^2+1}\,\dfrac{\cos\theta}{\sin\theta}=\dfrac{\sqrt{u^2+1}\,\sqrt{u^2+1-C^2}}{C}$

即 $$\frac{du}{\sqrt{u^2+1}\cdot\sqrt{u^2+1-C^2}}=\frac{1}{C}dv$$

故 $$v=C\int\frac{du}{\sqrt{u^2+1}\cdot\sqrt{u^2+1-C^2}}+\tilde{\tilde{C}}$$

八、已知 $\gamma(s)$ 是 $\mathbb{R}^3$ 中以弧长为参数的光滑闭曲线，曲率 $k(s)>0$，若其主法向量在球面上轨迹是简单闭曲线，并把球面分成两部分，求这两部分的面积比.

解. 设 $\{\gamma(s);\vec{t}(s),\vec{n}(s),\vec{b}(s)\}$ 是 $\gamma(s)$ 的 Frenet 标架，设 $\rho$ 是主法向量 $\vec{n}(s)$ 的弧长参数，因 $\frac{d}{ds}\vec{n}(s)=-k(s)\vec{t}(s)+\tau(s)\vec{b}(s)$

则 $\left|\dot{\vec{n}}(s)\right|^2=k^2(s)+\tau^2(s)$，于是 $\frac{d\rho}{ds}=\sqrt{k^2+\tau^2}$.

考虑 $\vec{n}(s)$ 在 $S^2$ 上围成的区域 D，注意到 $S^2$ 的 Gauss 曲率 $K=1$，

记 $k_g$ 是 $\vec{n}(s)$ 在 $S^2$ 上曲线的测地曲率，则由 Gauss-Bonnet 公式：

$$\iint_D 1\,dA+\int_{\partial D}k_g\,d\rho=2\pi.$$

下面计算 $\vec{n}(s)$ 的测地曲率 $k_g=\left\langle\frac{d^2\vec{n}}{d\rho^2},\ \underbrace{\vec{n}(\rho)}_{\text{球面 } S^2 \text{ 在 } \mathbb{R}^3 \text{ 中单位法向量}}\wedge\frac{d\vec{n}}{d\rho}\right\rangle$

$$=\left(\vec{n}(\rho),\frac{d\vec{n}}{d\rho},\frac{d^2\vec{n}}{d\rho^2}\right)$$

注意到：$\frac{d\vec{n}}{d\rho}=\frac{d\vec{n}}{ds}\frac{ds}{d\rho}$

$$\frac{d^2\vec{n}}{d\rho^2}=\frac{d^2\vec{n}}{ds^2}\left(\frac{ds}{d\rho}\right)^2+\frac{d\vec{n}}{ds}\frac{d^2s}{d\rho^2}$$

则 $k_g=\left(\vec{n}(\rho),\frac{d\vec{n}}{ds},\frac{d^2\vec{n}}{ds^2}\right)\left(\frac{ds}{d\rho}\right)^3$

由 Frenet 公式 $\frac{d\vec{n}}{ds}=-k\vec{t}+\tau\vec{b}\Rightarrow\frac{d^2\vec{n}}{ds^2}=-\dot{k}\vec{t}-k\tau\vec{n}+\dot{\tau}\vec{b}-\tau^2\vec{n}$

$$=-\dot{k}-(k\tau+\tau^2)\vec{n}+\dot{\tau}\vec{b}$$

故 $\left(\vec{n},\frac{d\vec{n}}{ds},\frac{d^2\vec{n}}{ds^2}\right)=\begin{vmatrix}0&1&0\\-k&0&\tau\\-\dot{k}&-(k\tau+\tau^2)&\dot{\tau}\end{vmatrix}=k\dot{\tau}-\tau\dot{k}$

进而：$k_g=(k\dot{\tau}-\tau\dot{k})\frac{1}{k^2+\tau^2}\left(\frac{ds}{d\rho}\right)=\frac{d}{ds}\left(\arctan\left(\frac{\tau}{k}\right)\right)\frac{ds}{d\rho}$

$$\Rightarrow \int_{\partial D} k_g \, ds = \int_{\partial D} d\left(\arctan\left(-\frac{\tau}{k}\right)\right) \underset{\uparrow}{=} 0$$

Stokes公式 $\partial(\partial D) = \phi$

故 $\text{Area}(D) = 2\pi = \frac{1}{2}\text{Area}(S^2) \quad \left(\text{Area}(S^2) = 4\pi\right)$

故球面二部分面积比为 $1:1$.