中国科学技术大学数学科学学院

2021年春季学期《微分方程II(H)》期中测验- 参考解答

2021年5月10日， 15:55 - 18:20, 二教2606

姓名: ____________ 学号: ______________

注意事项:

1. 请将解答写在答题纸上，试卷和答题纸一并上交;
2. 闭卷考试，总分110分. 得分超过100分时，成绩取整为100分;
3. 在试卷正文中，我们始终假定$\Omega \subset \mathbb{R}^n$ 为有界集且具有C^1的边界$\partial\Omega$.

试卷正文

1. [15 分] Recall that for $u \in C^1(\bar{\Omega})$ the Gauss-Green theorem states that

$$\int_\Omega \frac{\partial u}{\partial x_i} dx = \int_{\partial\Omega} u(\nu \cdot e_i) d\sigma, \tag{1}$$

where ν denotes the unit outward normal to $\partial\Omega$, e_i denotes the ith coordinate vector of $\mathbb{R}^n$ and $d\sigma$ is the area element on $\partial\Omega$.

(a) Show that for any $u \in W^{1,p}(\Omega), v \in W^{1,q}(\Omega)$, where $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have the following Green's formula

$$\int_\Omega u \frac{\partial v}{\partial x_i} dx = -\int_\Omega \frac{\partial u}{\partial x_i} v dx + \int_{\partial\Omega} uv(\nu \cdot e_i) d\sigma \tag{2}$$

where the value of u, v on the boundary $\partial\Omega$ is viewed as the Trace of u and v.

(b) Similarly, we have

$$\int_\Omega u \Delta v dx = -\int_\Omega \langle Du, Dv \rangle dx + \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} d\sigma \tag{3}$$

for any $u, v \in H^2(\Omega)$.

Solution.

(a) Firstly, for $u, v \in C^1(\bar{\Omega})$, we apply (1) to uv and obtain (2). Then for $u \in W^{1,p}(\Omega), v \in W^{1,q}(\Omega)$, there exist $u_m, v_\ell \in C^\infty(\bar{\Omega})$ such that

$$\begin{aligned} u_m &\to u, \quad \text{in } W^{1,p}(\Omega), \\ v_\ell &\to v, \quad \text{in } W^{1,q}(\Omega). \end{aligned}$$

For each $u_m, v_\ell \in C^\infty(\bar{\Omega})$, there holds

$$\int_\Omega u_m \frac{\partial v_\ell}{\partial x_i} dx = -\int_\Omega \frac{\partial u_m}{\partial x_i} v_\ell dx + \int_{\partial\Omega} u_m v_\ell (\nu \cdot e_i) d\sigma. \tag{4}$$

Let $m, \ell \to \infty$. We have

$$\left| \int_\Omega \frac{\partial u_m}{\partial x_i} v_\ell dx - \int_\Omega \frac{\partial u}{\partial x_i} v dx \right| \le \int_\Omega \left| \frac{\partial u_m}{\partial x_i} - \frac{\partial u}{\partial x_i} \right| |v_\ell| dx + \int_\Omega \left| \frac{\partial u}{\partial x_i} (v_\ell - v) \right| dx$$
$$\le \|u_{m,x_i} - u_{x_i}\|_{L^p(\Omega)} \|v_\ell\|_{L^q(\Omega)} + \|u_{x_i}\|_{L^p(\Omega)} \|v_\ell - v\|_{L^q(\Omega)}$$
$$\to 0, \qquad \text{as } m, \ell \to \infty.$$

Similarly,

$$\int_\Omega u_m \frac{\partial v_\ell}{\partial x_i} dx \quad \to \quad \int_\Omega u \frac{\partial v}{\partial x_i} dx$$

as $m, \ell \to \infty$. For the last term of (4), by Hölder inequality and Trace inequality

$$\left| \int_{\partial\Omega} u_m v_\ell (\nu \cdot e_i) d\sigma - \int_{\partial\Omega} uv(\nu \cdot e_i) d\sigma \right| \le \int_{\partial\Omega} |u_m v_\ell - uv| d\sigma$$
$$\le \int_{\partial\Omega} \left(|u_m||v_\ell - v| + |u_m - u||v| \right) d\sigma$$
$$\le \|Tu_m\|_{L^p(\partial\Omega)} \|Tv_\ell - Tv\|_{L^q(\partial\Omega)} + \|Tu_m - Tu\|_{L^p(\partial\Omega)} \|Tv\|_{L^q(\partial\Omega)}$$
$$\le C\|u_m\|_{W^{1,p}(\Omega)} \|v_\ell - v\|_{W^{1,q}(\Omega)} + C\|u_m - u\|_{W^{1,p}(\Omega)} \|v\|_{W^{1,q}(\Omega)}$$
$$\to 0$$

as $m, \ell \to \infty$. Therefore, letting $m, \ell \to \infty$ in (4) we obtain (2) for $u \in W^{1,p}(\Omega), v \in W^{1,q}(\Omega)$.

(b) For $u, v \in H^2(\Omega)$, we have $v_{x_i} = \frac{\partial v}{\partial x_i} \in H^1(\Omega)$. Then the trace of v_{x_i} on $\partial\Omega$ is well defined and belongs to $L^2(\partial\Omega)$. Replacing v by v_{x_i} in (2), we have

$$\int_\Omega u \frac{\partial^2 v}{\partial x_i^2} dx = -\int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx + \int_{\partial\Omega} u \frac{\partial v}{\partial x_i} (\nu \cdot e_i) d\sigma. \tag{5}$$

Then (3) follows from taking the sum of i from 1 to n in (5) and noting that

$$\frac{\partial v}{\partial \nu} = \langle Dv, \nu \rangle = \langle \sum_{i=1}^n \frac{\partial v}{\partial x_i} e_i, \nu \rangle.$$

2. [15 分] For $2 \le p < \infty$, apply the Green's formula (2) to prove the interpolation inequality

$$\|Du\|^2_{L^p(\Omega)} \le C\|u\|_{L^p(\Omega)} \|D^2 u\|_{L^p(\Omega)} \tag{6}$$

for $u \in W_0^{2,p}(\Omega)$, where $C > 0$ is a constant independent of u.

Solution. For $u \in W_0^{2,p}(\Omega)$, where $2 \le p < \infty$, we have $Du \in L^p(\Omega)$ and $Du|Du|^{p-2} \in L^q(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Apply Green's formula (2), we have

$$\int_\Omega |Du|^p dx = \sum_{i=1}^n \int_\Omega \langle u_{x_i}, u_{x_i} |Du|^{p-2} \rangle dx$$
$$= -\sum_{i=1}^n \int_\Omega u D_{x_i} \left(u_{x_i} |Du|^{p-2} \right) dx + \int_{\partial\Omega} u u_{x_i} |Du|^{p-2} (\nu \cdot e_i) d\sigma. \tag{7}$$

The second term on the right hand side of (7) vanishes since $u \in W_0^{2,p}(\Omega)$ implies that the Trace of u vanishes on the boundary $\partial\Omega$ and the trace inequality implies

$$\begin{aligned}\left|\int_{\partial\Omega} u u_{x_i} |Du|^{p-2}(\nu\cdot e_i)d\sigma\right| \le& \int_{\partial\Omega}|u||Du|^{p-1}d\sigma\\ \le& \|Tu\|_{L^p(\partial\Omega)}\|T(Du)\|^{p-1}_{L^p(\partial\Omega)}\\ \le& C\|Tu\|_{L^p(\partial\Omega)}\|u\|^{p-1}_{W^{2,p}(\Omega)} \ = \ 0. \end{aligned} \tag{8}$$

On the other hand, for $u \in W_0^{2,p}(\Omega)$ we have （similar with Evans 书第5章习题18）

$$\begin{aligned}\sum_{i=1}^n D_{x_i}\left(u_{x_i}|Du|^{p-2}\right) =& \Delta u|Du|^{p-2} + (p-2)\sum_{i=1}^n u_{x_i}|Du|^{p-3}D_{x_i}|Du|\\ =& \Delta u|Du|^{p-2} + (p-2)\sum_{i=1}^n u_{x_i}|Du|^{p-4}\langle Du, D_{x_i}Du\rangle \end{aligned} \tag{9}$$

Using (8), (9) and applying the generalized Hölder inequality to (7) imply that

$$\begin{aligned}\int_\Omega |Du|^p dx \le& C\int_\Omega |u||D^2u||Du|^{p-2}dx\\ \le& C\left(\int_\Omega |u|^p dx\right)^{1/p}\left(\int_\Omega |Du|^p dx\right)^{1/p}\left(\int_\Omega |Du|^p dx\right)^{\frac{p-2}{p}}\end{aligned}$$

which is equivalent to (6).

3. [15 分] Show that $H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n)$.

Solution. Let $\xi \in C^\infty(\mathbb{R}_+)$ be a cut-off function satisfying (see e.g., §5.5 in Evans)

$$\xi(t) \equiv 1 \quad \text{if } 0 \le t \le 1, \qquad \xi(t) = 0 \text{ if } t \ge 2, \qquad 0 \le \xi \le 1$$

and the derivative $|\xi'(t)| \le C$. Let $u \in H^1(\mathbb{R}^n)$.

Step 1. We consider $u^{(R)}(x) = u(x)\xi(|x|/R)$, which vanishes on $\{|x| > 2R\}$ and still belongs to $H^1(\mathbb{R}^n)$. Leibniz's formula implies

$$u^{(R)}_{x_i}(x) = u_{x_i}(x)\xi(\frac{|x|}{R}) + u(x)\xi'(\frac{|x|}{R})\frac{x_i}{|x|R}.$$

$$\begin{aligned}\|u^{(R)} - u\|^2_{H^1(\mathbb{R}^n)} =& \int_{\mathbb{R}^n}|u^{(R)} - u|^2dx + \sum_{i=1}^n\int_{\mathbb{R}^n}|u^{(R)}_{x_i} - u_{x_i}|^2dx\\ =& \int_{|x|>R}|u^{(R)} - u|^2dx + \sum_{i=1}^n\int_{|x|>R}|u^{(R)}_{x_i} - u_{x_i}|^2dx\\ \le& C\int_{|x|>R}\left(|u|^2 + |Du|^2\right)dx \ \to \ 0\end{aligned}$$

as $R \to \infty$ since $u \in H^1(\mathbb{R}^n)$.

Step 2. Next, we consider mollification of $u^{(R)}(x)$:

$$u^{(R)}_\varepsilon(x) = [\eta_\varepsilon * u^{(R)}](x),$$

where η_ε is the mollifier (see Appendix C.5 of Evans book). Then 定理1 in §5.3 of Evans book implies that $u^{(R)}_\varepsilon(x) \in C_c^\infty(\mathbb{R}^n)$ and converges to $u^{(R)}(x)$ as $\varepsilon \to 0$ in $H^1(V)$ for

any compact subset $V \subset\subset \mathbb{R}^n$.
For any small $\delta > 0$, we first choose a large $k \in \mathbb{N}$ such that $\|u^{(k)} - u\|_{H^1} < \delta/2$, then choose a small $\varepsilon = \varepsilon(k)$ such that $\|u^{(k)}_{\varepsilon(k)} - u^{(k)}\|_{H^1} < \delta/2$. Then $u_k(x) := u^{(k)}_{\varepsilon(k)}(x) \in C_c^\infty(\mathbb{R}^n)$ is a sequence of functions in $C_c^\infty(\mathbb{R}^n)$ converging to $u \in H^1(\mathbb{R}^n)$. By definition, this means that $H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n)$.
注：也可先做磨光，再取截断，即Step 2 与Step 1 次序可交换.

4. [20 分]

(a) 叙述Sobolev inequality for $p > n$.

(b) 叙述Rellich-Kondrachov Compactness Theorem 定理内容

(c) Applying the fact $W^{1,p}(\Omega) \subset\subset L^p(\Omega), 1 \le p \le \infty$ to show $W^{2,p}(\Omega) \subset\subset W^{1,p}(\Omega)$.

Solution.

(a) Let Ω be a bounded open set with a C^1 boundary $\partial\Omega$. Assume $u \in W^{1,p}(\Omega)$ for $n > p$. Then u has a continuous version, still denoted by u, satisfying

$$\|u\|_{C^{0,\gamma}(\bar{\Omega})} \le C\|u\|_{W^{1,p}(\Omega)}$$

for some constant $C = C(n, p, \Omega)$, where $\gamma = 1 - n/p$.

(b) Let Ω be a bounded open set with a C^1 boundary $\partial\Omega$. Suppose $1 \le p < n$. Then $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$ for each $1 \le q < p^*$.

(c) Let $\{u_m\}_{m=1}^\infty$ be a bounded sequence in $W^{2,p}(\Omega)$. We need to show that a subsequence $\{u_{m_j}\}_{j=1}^\infty \subset \{u_m\}_{m=1}^\infty$ converges to a function u in $W^{1,p}(\Omega)$. Since both $\{u_m\}_{m=1}^\infty$ and $\{D^\alpha u_m\}_{m=1}^\infty$ are bounded in $W^{1,p}(\Omega)$, where α is a multi-index with $|\alpha| = 1$, the compactness of $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$ implies that, after passing to a subsequence,

$$\begin{aligned} u_m &\to u, \qquad \text{in } L^p(\Omega) \\ D^\alpha u_m &\to v_\alpha, \qquad \text{in } L^p(\Omega) \end{aligned}$$

as $m \to \infty$. We claim that v_α is the weak derivative of u. Indeed, for each test function $\phi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega u_m D^\alpha \phi dx = -\int_\Omega D^\alpha u_m \phi$$

for each $m = 1, 2, \cdots$. Letting $m \to \infty$ yields that

$$\int_\Omega u D^\alpha \phi dx = -\int_\Omega v_\alpha \phi.$$

So we conclude that $v_\alpha = D^\alpha u$ is the weak derivative of u. In particular, we have that (after passing to a subsequence) $u_m \to u$ in $W^{1,p}(\Omega)$.

5. [15 分] Show that there exists a constant $C > 0$ depending on Ω, n and $1 \le p < \infty$ such that

$$\|u\|_{W^{1,p}(\Omega)} \le C\left(\|Du\|_{L^p(\Omega)} + \|Tu\|_{L^p(\partial\Omega)}\right)$$

for any $u \in W^{1,p}(\Omega)$, where $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ denotes the trace operator.

Solution. We argue by contradiction. Were the stated estimate false, for each integer $k = 1, \cdots$, there would exist a function $u_k \in W^{1,p}(\Omega)$ satisfying

$$\|u_k\|_{W^{1,p}(\Omega)} \geq k \left(\|Du_k\|_{L^p(\Omega)} + \|Tu_k\|_{L^p(\partial\Omega)}\right).$$

We normalize u_k by considering $v_k = u_k(\|u_k\|_{W^{1,p}(\Omega)})^{-1}$. Then

$$\|v_k\|_{W^{1,p}(\Omega)} = 1, \quad \|Dv_k\|_{L^p(\Omega)} \leq 1/k \to 0, \quad \|Tv_k\|_{L^p(\partial\Omega)} \leq 1/k \to 0 \tag{10}$$

Step 1. The compactness of $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$ implies that there exists a subsequence $\{v_{k_j}\}_{j=1}^{\infty}$ of $\{v_k\}_{k=1}^{\infty}$ and a function $v \in L^p(\Omega)$ such that

$$v_{k_j} \to v \qquad \text{in } L^p(\Omega).$$

Moreover, for each $\phi \in C_c^\infty(\Omega)$,

$$\int_\Omega v\phi_{x_i}dx = \lim_{k_j\to\infty}\int_\Omega v_{k_j}\phi_{x_i}dx = -\lim_{k_j\to\infty}\int_\Omega v_{k_j,x_i}\phi dx = 0.$$

Consequently $v \in W^{1,p}(\Omega)$ with $Dv = 0$ a.e. and

$$\begin{aligned}\|v\|^p_{L^p(\Omega)} &= \lim_{k_j\to\infty}\|v_{k_j}\|^p_{L^p(\Omega)}\\ &= \lim_{k_j\to\infty}\left(\|v_k\|^p_{W^{1,p}(\Omega)} - \int_\Omega |Dv_k|^p dx\right) = 1.\end{aligned}$$

Step 2. On the other hand,

$$\begin{aligned}\|Tv - Tv_{k_j}\|^p_{L^p(\partial\Omega)} \leq& C\|v - v_{k_j}\|^p_{W^{1,p}(\Omega)}\\ =& C\left(\|v - v_{k_j}\|^p_{L^p(\Omega)} + \|Dv - Dv_{k_j}\|^p_{L^p(\Omega)}\right)\\ =& C\left(\|v - v_{k_j}\|^p_{L^p(\Omega)} + \|Dv_{k_j}\|^p_{L^p(\Omega)}\right)\\ \to& 0, \quad \text{as} \quad k_j \to \infty.\end{aligned}$$

This together with the third inequality of (10) implies that $Tv = 0$. Since $v \in W^{1,p}(\Omega)$ and $\partial\Omega$ is C^1, $Tv = 0$ implies that $v \in W_0^{1,p}(\Omega)$. Then by Poincaré inequality we have $\|v\|_{L^p(\Omega)} \leq C\|Dv\|_{L^p(\Omega)} = 0$, contradicting with $\|v\|_{L^p(\Omega)} = 1$.

6. [15 分] Consider the bilinear form

$$B[u,v] = \int_\Omega \left(\sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^n b^i u_{x_i}v + cuv\right)dx$$

for $u, v \in H_0^1(\Omega)$, where $a^{ij}, b^i, c \in L^\infty(\Omega)$, $a^{ij} = a^{ji}$ and $(a^{ij}(x)) \geq \theta I > 0$ a.e. $x \in \Omega$. Prove that there exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

$$|B[u,v]| \leq \alpha\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}, \tag{11}$$
$$\beta\|u\|^2_{H_0^1(\Omega)} \leq B[u,u] + \gamma\|u\|^2_{L^2(\Omega)}. \tag{12}$$

for all $u, v \in H_0^1(\Omega)$.

Solution. See Theorem 2 in §6.2 of Evans book

7. [15 分] Let

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + c(x)u, \tag{13}$$

where $a^{ij}, c \in L^\infty(\Omega), a^{ij} = a^{ji}$ and $(a^{ij}(x)) \geq \theta I > 0$ a.e. $x \in \Omega$.

(a) Show that there exists a constant $\mu \geq 0$ such that for each $f \in H^{-1}(\Omega)$ and $g \in H^1(\Omega)$, the boundary-value problem

$$\begin{cases} Lu = & f \quad \text{in } \Omega \\ \ \ u = & g \quad \text{on } \Omega, \end{cases} \tag{14}$$

has a unique weak solution $u \in H^1(\Omega)$, provided that $c(x) \geq -\mu,\ x \in \Omega$.

(b) Show that the solution $u \in H^1(\Omega)$ in (a) satisfies

$$\|u\|_{H^1(\Omega)} \leq C\left(\|g\|_{H^1(\Omega)} + \|f\|_{H^{-1}(\Omega)}\right).$$

Solution.

(a) Step 1. Let

$$B[u,v] = \int_\Omega \left(\sum_{i,j=1}^{n} a^{ij}u_{x_i}v_{x_j} + cuv\right) dx$$

be the bilinear form w.r.t the operator (13). From the proof of (12), we see that

$$\theta\|Du\|_{L^2}^2 \leq B[u,u] - \int_\Omega c(x)u^2 dx$$

If $c(x) \geq 0$, then $\theta\|Du\|_{L^2}^2 \leq B[u,u]$. If $\mu_0 = -\inf_{x\in\Omega} c(x) > 0$, then

$$\begin{aligned} \theta\|Du\|_{L^2}^2 \leq & B[u,u] - \int_\Omega c(x)u^2 dx \\ \leq & B[u,u] + \mu_0\|u\|_{L^2}^2 \\ \leq & B[u,u] + \mu_0 c_0\|Du\|_{L^2}^2, \end{aligned}$$

where we used the Poincaré inequality

$$\|u\|_{L^2(\Omega)} \leq c_0\|Du\|_{L^2(\Omega)} \tag{15}$$

for $u \in H_0^1(\Omega)$. When μ_0 satisfies $\theta > \mu_0 c_0$, then

$$B[u,u] \ \geq \ (\theta - \mu_0 c_0)\|Du\|_{L^2}^2 \ \geq \ \frac{\theta - \mu_0 c_0}{1 + c_0}\|u\|_{H_0^1(\Omega)}^2.$$

Therefore, for any constant μ satisfying $\theta > \mu c_0$, the bilinear form $B[u,v]$ satisfies the condition of Lax-Milgram theorem, provided that $c(x) \geq -\mu$.

Step 2. Let $\tilde{u} = u - g$. The problem (14) is equivalent to

$$\begin{cases} L\tilde{u} = f - Lg & \text{in } \Omega \\ \ \ \tilde{u} = 0 & \text{on } \Omega. \end{cases} \tag{16}$$

Note that $f - Lg \in H^{-1}(\Omega)$. The Lax-Milgram theorem implies that there exists a unique weak solution $\tilde{u} \in H_0^1(\Omega)$ to the problem (16). In particular, $u = \tilde{u} + g \in H^1(\Omega)$ is the unique weak solution to the problem (14).

(b) From the energy estimate in (a),

$$\begin{aligned}\frac{\theta-\mu c_0}{1+c_0}\|\tilde{u}\|_{H^1}^2 \leq & B[\tilde{u},\tilde{u}] = \langle f - Lg, \tilde{u}\rangle \\ \leq & \|f-Lg\|_{H^{-1}(\Omega)}\|\tilde{u}\|_{H_0^1} \\ \leq & C\left(\|f\|_{H^{-1}} + \|g\|_{H^1}\right)\|\tilde{u}\|_{H_0^1}.\end{aligned}$$

This implies that

$$\|u\|_{H^1} = \|\tilde{u}+g\|_{H^1} \leq C\left(\|f\|_{H^{-1}} + \|g\|_{H^1}\right).$$