2020-2021年第三次小测解答.

1、计算下列各题.

(a) 设Γ为椭圆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a,b>0)$ 在第二象限的部分.

计算 $\int_\Gamma xy\,ds$.

解：这是第一型曲线积分，参数形式为 $\begin{cases} x=a\cos\theta \\ y=b\sin\theta \end{cases}$ $\theta\in[\frac{\pi}{2},\pi]$

$$r'(\theta)=(x'(\theta),y'(\theta))=(-a\sin\theta,\ b\cos\theta)$$

则原式 $=\int_{\frac{\pi}{2}}^{\pi} ab\sin\theta\cos\theta\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta$

$$=\int_{\frac{\pi}{2}}^{\pi} ab\sin\theta\cos\theta\sqrt{b^2+(a^2-b^2)\sin^2\theta}\,d\theta$$

$$=\int_{\frac{\pi}{2}}^{\pi}\frac{1}{2}ab\sqrt{b^2+(a^2-b^2)\sin^2\theta}\,d\sin^2\theta.$$

$$=-\frac{1}{2}ab\int_0^1\sqrt{b^2+(a^2-b^2)t}\,dt.$$

$$=-\frac{1}{2}ab\times\frac{2}{3}\left[b^2+(a^2-b^2)t\right]^{\frac{3}{2}}\Big|_0^1\times\frac{1}{a^2-b^2}$$

$$=-\frac{ab}{3}(a^3-b^3)\times\frac{1}{a^2-b^2}.$$

$$=-\frac{ab(a^3-b^3)}{3(a^2-b^2)}.$$

(b) Γ是球面 $x^2+y^2+z^2=1$ 与平面 $x+y+z=0$ 的圆周，从第一卦限看，Γ逆时针方向为正向。计算 $\int_\Gamma dx+ydy$.

解：$\int_\Gamma dx+ydy=\int_\Gamma dx+ydy+0dz.$

视为R^3中的第二型曲线积分，$P=1$，$Q=y$，$R=0$

则由 Stokes 公式：

原式 $=\iint_\Omega(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})dydz+(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})dzdx+(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy$

$=0.$

(c) Σ 是第一卦限中的球面 $x^2+y^2+z^2=a^2$，计算 $\int_\Sigma x^2 d\sigma$.

解：这是一个第一型曲面积分.

曲面参数表达为 $\begin{cases} x=a\sin\theta\cos\varphi \\ y=a\sin\theta\sin\varphi \\ z=a\cos\theta \end{cases}$，$\theta\in[0,\pi]$，$\varphi\in[0,2\pi]$

则 $\|r_\theta\times r_\varphi\|=a^2\sin\theta.$

代入 $A\cdot 8=\iint_\Delta a^2\sin^2\theta\cos^2\varphi\, a^2\sin\theta\, d\theta d\varphi.$

$=\iint_\Delta a^4\sin^3\theta\cos^2\varphi\, d\theta d\varphi.$

$=a^4\int_0^\pi \sin^3\theta d\theta\int_0^{2\pi}\cos^2\varphi\, d\varphi.$

$=a^4\times 2\times\frac{2!!}{3!!}\times\pi.$

$=\frac{4}{3}\pi a^4.$

则积分值为 $A=\frac{1}{6}\pi a^4.$

(d) Σ 为椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 的外侧，计算积分 $\iint_\Sigma dydz+ydzdx+y^2dxdy.$

① 由Gauss公式：$\iint_\Sigma dydz+ydzdx+y^2dxdy$

$=\iiint_\Omega\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dxdydz$

$=\iiint_\Omega 1\,dxdydz$

$=\frac{4}{3}\pi abc.$

② 椭球面的参数形式为 $\begin{cases} x=a\sin\theta\cos\varphi \\ y=b\sin\theta\sin\varphi \\ z=c\cos\theta \end{cases}$，$\theta\in[0,\pi]$，$\varphi\in[0,2\pi]$

$\frac{\partial x}{\partial\theta}=a\cos\theta\cos\varphi.$ $\qquad\frac{\partial x}{\partial\varphi}=-a\sin\theta\sin\varphi.$

$$\frac{\partial y}{\partial \theta} = b\cos\theta\sin\varphi \qquad \frac{\partial y}{\partial \varphi} = b\sin\theta\cos\varphi$$

$$\frac{\partial z}{\partial \theta} = -c\sin\theta \qquad \frac{\partial z}{\partial \varphi} = 0$$

则 $r_\theta \times r_\varphi$ 应与椭圆外侧相符，代入得

$$原式 = \iint\limits_{[0,\pi]\times[0,2\pi]} \frac{\partial(y,z)}{\partial(\theta,\varphi)} + b\sin\theta\sin\varphi\frac{\partial(z,x)}{\partial(\theta,\varphi)} + b^2\sin^2\theta\sin^2\varphi\frac{\partial(x,y)}{\partial(\theta,\varphi)}d\theta d\varphi$$

$$= \iint\limits_{\Delta} bc\sin^2\theta\cos\varphi + abc\sin^3\theta\sin^2\varphi + b^2\sin^2\theta\sin^2\varphi \times ab\sin\theta\cos\theta d\theta d\varphi$$

$$= \iint\limits_{\Delta} bc\sin^2\theta\cos\varphi + abc\sin^3\theta\sin^2\varphi + ab^3\sin^3\theta\cos\theta\sin^2\varphi d\theta d\varphi$$

$$= \iint\limits_{\Delta} abc\sin^3\theta\sin^2\varphi d\theta d\varphi$$

$$= \pi abc\int_0^{\pi}\sin^3\theta d\theta$$

$$= 2\pi abc\int_0^{\frac{\pi}{2}}\sin^3\theta d\theta$$

$$= 2\pi abc \times \frac{2!!}{3!!}$$

$$= \frac{4}{3}\pi abc.$$

结果相同，均为 $\frac{4}{3}\pi abc$.

(e) 计算 $\int_\Gamma ydx + zdy + xdz$，其中 Γ 是平面 $x+y=2$ 和球面 $x^2+y^2+z^2=2(x+y)$ 交成的圆周，从原点看去，顺时针方向是 Γ 的正向.

解：Γ 是简单封闭曲线，$P=y$，$Q=z$，$R=x$.

$$原式 = \iint\limits_{\Sigma}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

$$= -\iint\limits_{\Sigma} dydz + dzdx + dxdy.$$

且 Σ 的法向量为 $\frac{1}{\sqrt{2}}(1,1,0)$

$$原式 = -\iint_{\Sigma}(1,1,1)\cdot\frac{1}{\sqrt{2}}(1,1,0)\,d\sigma$$

$$= -\iint_{\Sigma}\sqrt{2}\,d\sigma$$

$$= -\sqrt{2}\times\pi(\sqrt{2})^2$$

$$= -2\sqrt{2}\pi.$$

2、设 $\vec{v}=(P,Q,R)$ 为向量场，各分量具有二阶连续偏导数，证明：

$$\nabla(\nabla\cdot\vec{v})-\nabla\times(\nabla\times\vec{v})=\Delta v$$

其中 $\Delta v=(\Delta P,\Delta Q,\Delta R)$.

证明：$\nabla\cdot\vec{v}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$,

$$\nabla\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)$$

$$=\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 Q}{\partial x\partial y}+\frac{\partial^2 R}{\partial x\partial z},\ \frac{\partial^2 P}{\partial x\partial y}+\frac{\partial^2 Q}{\partial y^2}+\frac{\partial^2 R}{\partial y\partial z},\ \frac{\partial^2 P}{\partial x\partial z}+\frac{\partial^2 Q}{\partial y\partial z}+\frac{\partial^2 R}{\partial z^2}\right)$$

$$\nabla\times\vec{v}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)$$

$\nabla\times(\nabla\times\vec{v})$ 第一个分量为 $\dfrac{\partial\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)}{\partial y}-\dfrac{\partial\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)}{\partial z}$.

$$=\frac{\partial^2 Q}{\partial x\partial y}-\frac{\partial^2 P}{\partial y^2}-\frac{\partial^2 P}{\partial z^2}+\frac{\partial^2 R}{\partial x\partial z}$$

同理、第二个分量为 $\frac{\partial^2 P}{\partial x\partial y}-\frac{\partial^2 Q}{\partial x^2}-\frac{\partial^2 Q}{\partial z^2}+\frac{\partial^2 R}{\partial y\partial z}$,

第三个分量为 $\frac{\partial^2 P}{\partial x\partial z}+\frac{\partial^2 Q}{\partial y\partial z}-\frac{\partial^2 R}{\partial x^2}-\frac{\partial^2 R}{\partial y^2}$.

相减即得到 $(\Delta P,\Delta Q,\Delta R)$，得证 $\blacksquare$

3、证明向量场 $v=((2x+y+z)yz,(x+2y+z)zx,(x+y+2z)xy)$ 是有势场，并求出它所有的势函数

证明：$P=(2x+y+z)yz$，$Q=(x+2y+z)zx$，$R=(x+y+2z)xy$.

V定义域为R^3，是单连通的，故只需证V是无旋场。

$$\frac{\partial R}{\partial y}=xy+x(x+y+2z),\quad \frac{\partial Q}{\partial z}=2x+(x+2y+z)x.$$

则$\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}=0$，其余分量同理也为0.

故$\vec{V}$是无旋场，也是有势场.

$$\varphi_0(a,b,c)=\int_{(0,0,0)}^{(a,b,c)} P\,dx+Q\,dy+R\,dz.$$

$$=\int_0^a\left(2t+\frac{b}{a}t+\frac{c}{a}t\right)\frac{bc}{a^2}t^2+\left(t+\frac{2b}{a}t+\frac{c}{a}t\right)\frac{bc}{a^2}t^2+\left(t+\frac{b}{a}t+\frac{2c}{a}t\right)\frac{bc}{a^2}t^2\,dt$$

$$=\int_0^a\left(4t+\frac{4b}{a}t+\frac{4c}{a}t\right)\frac{bc}{a^2}t^2\,dt.$$

$$=\frac{4bc}{a^3}\int_0^a(a+b+c)t^3\,dt.$$

$$=\frac{4bc}{a^3}\times(a+b+c)\times\frac{a^4}{4}$$

$$=abc(a+b+c)$$

即$\varphi_0(x,y,z)=xyz(x+y+z)$.

所有的势函数为$\varphi(x,y,z)=xyz(x+y+z)+C$.

4. 设在R^3空间中光滑曲线Σ围成闭区域Ω，外法向$\vec{n}$指向S的正侧，函数u,v在Ω上具有二阶的连续偏导数，证明a–d：

(a) $$\iiint_\Omega v\Delta u\,dxdydz=-\iiint_\Omega \nabla u\cdot\nabla v\,dxdydz+\int_\Sigma v\frac{\partial u}{\partial \vec{n}}\,d\sigma$$

证明：$$\int_\Sigma v\frac{\partial u}{\partial \vec{n}}\,d\sigma=\int_\Sigma v(\nabla u\cdot\vec{n})\,d\sigma=\int_\Sigma(v\nabla u)\cdot\vec{n}\,d\sigma$$

$$v\nabla u=\left(v\frac{\partial u}{\partial x},v\frac{\partial u}{\partial y},v\frac{\partial u}{\partial z}\right)$$

则$\int_\Sigma(v\nabla u)\cdot\vec{n}\,d\sigma$

$$=\int_{\Sigma} v\frac{\partial u}{\partial x}dydz+v\frac{\partial u}{\partial y}dzdx+v\frac{\partial u}{\partial z}dxdy$$

$$=\iiint_{\Omega}\frac{\partial(v\frac{\partial u}{\partial x})}{\partial x}+\frac{\partial(v\frac{\partial u}{\partial y})}{\partial y}+\frac{\partial(v\frac{\partial u}{\partial z})}{\partial z}dxdydz.$$

$$=\iiint_{\Omega}\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}+v\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial y^2}+\frac{\partial v}{\partial z}\frac{\partial u}{\partial z}+v\frac{\partial^2 u}{\partial z^2}dxdydz$$

$$=\iiint_{\Omega}\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)\cdot\left(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y},\frac{\partial v}{\partial z}\right)+v\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)dxdydz$$

$$=\iiint_{\Omega}\nabla u\cdot\nabla v\,dxdydz+\iiint_{\Omega}v\Delta u\,dxdydz. \quad \square$$

(b) $\iiint_{\Omega}(v\Delta u-u\Delta v)dxdydz=\int_{\Sigma}(v\frac{\partial u}{\partial \vec{n}}-u\frac{\partial v}{\partial \vec{n}})d\sigma.$

由(a): $\begin{cases}\iiint_{\Omega}v\Delta u\,dxdydz=-\iiint_{\Omega}\nabla u\cdot\nabla v\,dxdydz+\int_{\Sigma}v\frac{\partial u}{\partial \vec{n}}d\sigma & ①\\ \iiint_{\Omega}u\Delta v\,dxdydz=-\iiint_{\Omega}\nabla v\cdot\nabla u\,dxdydz+\int_{\Sigma}u\frac{\partial v}{\partial \vec{n}}d\sigma & ②\end{cases}$

①−②，即 $\iiint_{\Omega}(v\Delta u-u\Delta v)dxdydz=\int_{\Sigma}(v\frac{\partial u}{\partial \vec{n}}-u\frac{\partial v}{\partial \vec{n}})d\sigma \quad \square$

(c) u在Ω中满足$\Delta u=0$(此时称u为Ω上的调和函数)

当且仅当对于Ω中任意简单闭曲面S有$\int_{S}\frac{\partial u}{\partial n}d\sigma=0.$

证明：在(a)中，代入$v=1$,

得 $\iiint_{\Omega}\Delta u\,dxdydz=\int_{S}\frac{\partial u}{\partial n}d\sigma.$

当$\Delta u\equiv 0$时，则$\int_{S}\frac{\partial u}{\partial \vec{n}}d\sigma=0.$

若对$\forall S$, $\int_{S}\frac{\partial u}{\partial \vec{n}}d\sigma=0$，则$\Delta u\equiv 0$，得证 $\square$.

(d) 若u为Ω上的调和函数，则u在Ω内部的值由它在Σ上的值唯一确定.

证明：先说明如下等式成立，

$$u(p_0)=\frac{1}{4\pi}\iint_{\Sigma}\left(u\frac{\cos(p,n)}{p^2}+\frac{1}{p}\frac{\partial u}{\partial \vec{n}}\right)d\sigma.$$

其中 $\vec{p}$ 是连接 Σ 上的点与 p_0 的向量，$p=\|\vec{p}\|$.

首先，$v=\frac{1}{p}$ 是 $R^3\setminus p_0$ 上的调和函数.

取 ε 足够小，$B_\varepsilon(p_0)\subset\Omega$.

由(b)：$\iiint_\Omega (v\Delta u-u\Delta v)dxdydz=\int_\Sigma (v\frac{\partial u}{\partial\vec{n}}-u\frac{\partial v}{\partial\vec{n}})d\sigma$.

则 $\iiint_{\Omega\setminus B_\varepsilon(p_0)} (v\Delta u-u\Delta v)dxdydz=0=\int_{\Omega\setminus B_\varepsilon(p_0)}(v\frac{\partial u}{\partial\vec{n}}-u\frac{\partial v}{\partial\vec{n}})d\sigma$.

则 $\int_\Omega (v\frac{\partial u}{\partial\vec{n}}-u\frac{\partial v}{\partial\vec{n}})d\sigma=\int_{B_\varepsilon(p_0)}(v\frac{\partial u}{\partial\vec{n}}-u\frac{\partial v}{\partial\vec{n}})d\sigma$.

$\nabla\vec{p}\cdot\vec{n}=\frac{\vec{p}\cdot\vec{n}}{\|\vec{p}\|}$

且 $\int_\Omega (v\frac{\partial u}{\partial\vec{n}}-u\frac{\partial v}{\partial\vec{n}})d\sigma$

$=\int_\Omega(\frac{1}{p}\frac{\partial u}{\partial\vec{n}}-u\nabla v\cdot\vec{n})d\sigma$

$=\int_\Omega(\frac{1}{p}\frac{\partial u}{\partial\vec{n}}+u\frac{\cos(p,n)}{p^2})d\sigma$

$=\int_{B_\varepsilon(p_0)}(\frac{1}{p}\frac{\partial u}{\partial\vec{n}}+u\frac{1}{p^2})d\sigma$

$=\frac{1}{\varepsilon^2}\int_{B_\varepsilon(p_0)}u\,d\sigma$

$=4\pi u|_\varepsilon$

当 $\varepsilon\to 0$ 时，$\int_{B_\varepsilon(p_0)}(\frac{1}{p}\frac{\partial u}{\partial\vec{n}}+u\frac{1}{p^2})d\sigma=4\pi u(p_0)$

即 $u(p_0)=\frac{1}{4\pi}\int_\Sigma(u\frac{\cos(p,n)}{p^2}+\frac{1}{p}\frac{\partial u}{\partial\vec{n}})d\sigma$.

故可看出 $u(p_0)$ 完全由 u 在 Σ 上的取值决定 $\square$.

(e) 若 u 为 Ω 上的调和函数，$(x_0,y_0,z_0)\in\Omega^\circ$，$B_r(x_0,y_0,z_0)$ 表示以 (x_0,y_0,z_0) 为球心，$r>0$ 为半径的球，其包含在 Ω 中.

则有：$u(x_0,y_0,z_0)=\frac{1}{4\pi r^2}\int_{\partial B_r(x_0,y_0,z_0)}u(x,y,z)d\sigma$.

证明：由(d)，$u(p_0)=\frac{1}{4\pi}\int_{\Sigma}\left(u\frac{\cos(p,n)}{p^2}+\frac{1}{p}\frac{\partial u}{\partial n}\right)d\sigma$

取 Σ 为 $\partial B_r(x_0,y_0,z_0)$.

则 $\vec{p}$ 与 $\vec{n}$ 平行，$\cos(p,n)=1$，$\|\vec{p}\|=r$.

则 $u(x_0,y_0,z_0)=\frac{1}{4\pi}\int_{\partial B_r(x_0,y_0,z_0)}\left(u\frac{1}{r^2}+\frac{1}{r}\frac{\partial u}{\partial \vec{n}}\right)d\sigma$

$$=\frac{1}{4\pi r^2}\int_{\partial B_r(x_0,y_0,z_0)}u(x,y,z)\,d\sigma+\frac{1}{4\pi r}\int_{\partial B_r(x_0,y_0,z_0)}\frac{\partial u}{\partial \vec{n}}\,d\sigma$$

$$=\frac{1}{4\pi r^2}\int_{\partial B_r(x_0,y_0,z_0)}u(x,y,z)\,d\sigma+\frac{1}{4\pi r}\iiint_{B_r(x_0,y_0,z_0)}\Delta u\,d\mu$$

$$=\frac{1}{4\pi r^2}\int_{\partial B_r(x_0,y_0,z_0)}u(x,y,z)\,d\sigma$$

即 $u(x_0,y_0,z_0)=\frac{1}{4\pi r^2}\int_{\partial B_r(x_0,y_0,z_0)}u(x,y,z)\,d\sigma$，得证 $\square$