2020-2021 数分A2 期末解答.

1. $\iint\limits_D \frac{2x}{xy^3+y^2}\,dx\,dy$. D为 $xy=1$, $xy=3$, $y^2=x$, $y^2=3x$

围成区域

解: $\begin{cases} u = xy & \frac{\partial u}{\partial x} = y \quad \frac{\partial u}{\partial y} = x \\ v = \frac{y^2}{x} & \frac{\partial v}{\partial x} = -\frac{y^2}{x^2}, \quad \frac{\partial v}{\partial y} = \frac{2y}{x} \end{cases}$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{3y^2}{x} = 3v$$

$$\frac{2x}{xy^3+y^2} = \frac{2x}{xv(1+xy)} = \frac{2}{v(1+xy)} = \frac{2}{(1+u)v}$$

原式 $= \iint\limits_{[1,3]^2} \frac{2}{(1+u)v} \times \frac{1}{3v}\,du\,dv$

$= \int_1^3 \frac{2}{3v^2}\,dv \int_1^3 \frac{1}{1+u}\,du$

$= \frac{2}{3}\left(-\frac{1}{v}\right)\Big|_1^3 \ln(1+u)\Big|_1^3$

$= \frac{2}{3} \times \frac{2}{3} \times \ln 2 = \frac{4}{9}\ln 2$.

2、 $u = u(x,y,z)$ 由 $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$

所确定的隐函数.

证: $|\mathrm{grad}\ u|^2 = 2\vec{r} \cdot \mathrm{grad}$. $\vec{r} = (x, y, z)$

$\mathrm{grad} = \nabla$

$\frac{\partial u}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\partial F \ldots} = +\frac{\frac{2x}{a^2+u}}{\ldots}$

$$\frac{\frac{2y}{a^2+u}}{\left(\frac{x^2}{a^2+u}\right)^2+\left(\frac{y^2}{b^2+u}\right)^2+\left(\frac{z^2}{c^2+u}\right)^2}, \quad |\nabla u|^2=\frac{4}{\left(\frac{x^2}{a^2+u}\right)^2+\left(\frac{y^2}{b^2+u}\right)^2+\left(\frac{z^2}{c^2+u}\right)^2}$$

则 $|\mathrm{grad}\, u|^2 = 2\vec{r}\cdot \mathrm{grad}$.

3、L是以O为起点，(a,b,c) 终点 $a,b,c>0$

$a^2+\frac{b^2}{3}+\frac{c^2}{6}=1$，则 a、b、c 取何值时。

$\int_L yz\,dx+zx\,dy+xy\,dz$ 最大。$=$?

$$\begin{cases} x=t \\ y=\frac{b}{a}t \\ z=\frac{c}{a}t \end{cases} \Rightarrow t: 0\to a.$$

$$原式=\int_0^a \frac{bc}{a^2}t^2+\frac{c}{a}t^2\times\frac{b}{a}+\frac{b}{a}t^2\times\frac{c}{a}\,dt$$

$$=\int_0^a \frac{3bc}{a^2}t^2\,dt$$

$$=abc$$

即 $a^2+\frac{b^2}{3}+\frac{c^2}{6}=1$ 时，abc 最大值。

即求 $a^2\left(\frac{b}{\sqrt{3}}\right)^2\left(\frac{c}{\sqrt{6}}\right)^2$ 最大

$x+y+z=1 \geq 3\sqrt[3]{xyz}$.

则 $xyz \le \frac{1}{27}$，$a\frac{b}{\sqrt{3}}\frac{c}{\sqrt{6}} \le \frac{1}{3\sqrt{3}}$.

$abc \le \frac{\sqrt{6}}{3}$

当且仅当 $a=\frac{b}{\sqrt{3}}=\frac{c}{\sqrt{6}}=\frac{1}{\sqrt{3}}$ 时，最大值取到

4. S：$x^2+y^2+z^2-yz=1$ 上的动点，P 在 S 上 S 在 P 处切平面与 Oxy 垂直，求 P 的 L 轨迹，求 $\iint_{\Sigma} \frac{(x+3)|y-2z|}{\sqrt{4+y^2+z^2-4yz}}d\sigma$，$\Sigma$ 是 S 上 L 的上部分.

解：$F(x,y,z)=x^2+y^2+z^2-yz-1=0$

$\frac{\partial F}{\partial x}=2x$，$\frac{\partial F}{\partial y}=2y-z$，$\frac{\partial F}{\partial z}=2z-y$.

则切平面为 $(x-x_0, y-y_0, z-z_0)\cdot(2x_0, 2y_0-z_0, 2z_0-y_0)=0$

即 $2x_0(x-x_0)+(2y_0-z_0)(y-y_0)+(2z_0-y_0)(z-z_0)=0$

$(0,0,1)\cdot(2x_0, 2y_0-z_0, 2z_0-y_0)=0$.

即 $2z_0=y_0$，轨迹为 $y=2z$

$\begin{cases} x^2+y^2+z^2-yz=1 \\ y=2z. \end{cases}$ 为 L 的表达式

$$\iint_{\Sigma} \frac{(x+3)|y-2z|}{\sqrt{4+y^2+z^2-4yz}}d\sigma=\iint_{\Sigma} \frac{(x+3)(2z-y)}{\sqrt{4+y^2+z^2-4yz}}d\sigma.$$

且 $\sqrt{4x_0^2+(2y_0-z_0)^2+(2z_0-y_0)^2}=\sqrt{4x_0^2+4y_0^2+4z_0^2+y_0^2+z^2-8y_0z_0}$

$=\sqrt{4+y_0^2+z_0^2-4y_0z_0}$，即为外法向的长度.

$\vec{n} = \frac{1}{\sqrt{4+y_0^2+z_0^2-4y_0z_0}}(2x_0, 2y_0-z_0, 2z_0-y_0)$

则记 $\vec{F} = (0, 0, x+3)$. $\vec{F}\cdot\vec{n} = \frac{(x+3)(2z_0-y_0)}{\sqrt{4+y_0^2+z_0^2-4y_0z_0}}$

则原式 $= \iint_{\Sigma} \vec{F}\cdot\vec{n}\, d\sigma = \iint_{\Sigma}(x+3)\,dx\,dy$

且 $\begin{cases} x^2+y^2+z^2-xy=1 \\ y=2z \end{cases}$ 代入得 $x^2+\frac{3}{4}y^2=1$.

原式 $= 3\iint_{D} dx\,dy = 3\times\pi\times1\times\frac{2}{\sqrt3} = 2\sqrt3\pi$.

5、二型曲面积分：$\iint_{\Sigma} 4xz\,dy\,dz - 2yz\,dz\,dx + (x^2-z^2)\,dx\,dy$

其中 Σ 由 $z=e^y$，$0\le y\le1$，绕 z 轴旋转一周
(外侧为正)

解：法一：将现有曲面补成封闭曲面.

Σ': $\begin{cases} x^2+y^2=1 \\ z=e \end{cases}$

$$\iint_{\Sigma+\Sigma'} 4xz\,dy\,dz - 2yz\,dz\,dx + (x^2-z^2)\,dx\,dy$$

$$= \iiint_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz = 0$$

且 $\iint_{\Sigma'} 4xz\,dy\,dz - 2yz\,dz\,dx + (x^2-z^2)\,dx\,dy$

$$= \iint_{\Sigma'}(x^2-z^2)\,dx\,dy = \iint_{\Sigma'} x^2\,dx\,dy - e^2\pi.$$

$\Big\{\ x = r\cos\theta$ ... $\left|\frac{\partial(x,y)}{}\right|$

$$\begin{cases} \\ y = r\sin\theta \end{cases} \quad \theta \in [0, 2\pi),\ r \in [0,1].\ \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r.$$

原式 $= \displaystyle\iint_{[0,1]\times[0,2\pi)} r^2\cos^2\theta \times r\, dr\, d\theta = \frac{1}{4}\pi.$

则 $\displaystyle\iint_{\Sigma} 4xz\,dydz - 2yz\,dzdx + (x^2 - z^2)\,dxdy = (e^2 - \frac{1}{4})\pi.$

法二：直接法：曲面的参数形式为 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = e^r \end{cases}$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial x}{\partial r} = \cos\theta$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta \qquad r_\theta \times r_r \text{ 是正向}$$

$$\frac{\partial z}{\partial \theta} = 0 \qquad \frac{\partial z}{\partial r} = e^r$$

原式 $= \displaystyle\iint_{[0,1]\times[0,2\pi)} 4re^r\cos\theta \frac{\partial(y,z)}{\partial(\theta,r)} - 2re^r\sin\theta \frac{\partial(z,x)}{\partial(\theta,r)} + (r^2\cos^2\theta - e^{2r}) \frac{\partial(x,y)}{\partial(\theta,r)}\, dr d\theta.$

$= \displaystyle\iint_{[0,1]\times[0,2\pi)} 4r^2e^{2r}\cos^2\theta - 2r^2e^{2r}\sin^2\theta - r^3\cos^2\theta + re^{2r}\, drd\theta$

$= \displaystyle\iint_{\Delta} 4r^2e^{2r}\cos^2\theta - 2r^2e^{2r} + 2r^2e^{2r}\cos^2\theta - r^3\cos^2\theta + re^{2r}\, drd\theta$

$= \displaystyle\iint_{\Delta} 6r^2e^{2r}\cos^2\theta - 2r^2e^{2r} - r^3\cos^2\theta + re^{2r}\, drd\theta.$

$= \displaystyle\int_0^1 dr \int_0^{2\pi} 6r^2e^{2r}\cos^2\theta - 2r^2e^{2r} - r^3\cos^2\theta + re^{2r}\, d\theta$

$$=\int_0^1 6\pi r^2 e^{2r} - 4\pi r^2 e^{2r} - \pi r^3 + 2\pi r e^{2r} dr$$

$$=\int_0^1 2\pi r^2 e^{2r} + 2\pi r e^{2r} - \pi r^3 dr$$

$$=2\pi\int_0^1 r^2 e^{2r} + r e^{2r} dr - \frac{1}{4}\pi$$

$$\int_0^1 r e^{2r} dr = \frac{1}{2}\int_0^1 e^{2r} dr^2$$

$$=\frac{1}{2}\left[(e^{2r} r^2)\Big|_0^1 - \int_0^1 r^2 de^{2r}\right]$$

$$=\frac{1}{2}\left(e^2 - 2\int_0^1 r^2 e^{2r} dr\right)$$

$$=\frac{1}{2}e^2 - \int_0^1 r^2 e^{2r} dr$$

则原式 $=(e^2-\frac{1}{4})\pi$

6、计算由三个圆柱体 $x^2+y^2\le a^2$，$x^2+z^2\le a^2$，$y^2+z^2\le a^2$ 交成的几何体体积

解：先记 $a=1$，$V=\iiint_\Omega 1\,dxdydz$

$$V=\iiint_\Omega 1\,dxdydz = 48\iiint_{\substack{x,y,z\ge 0\\ x\ge y\ge z\\ x^2+y^2\le a^2}} 1\,dxdydz$$

$y\le\sqrt{a^2-x^2}$ 与 $y\le x$

$$=48\int_0^a dx\int_0^x dy\int_0^y dz$$

$$=48\int_0^{\frac{\sqrt{2}}{2}a} dx\int^x dy\int^y dz + 48\int_{\frac{\sqrt{2}}{2}a}^a dx\int^{\sqrt{a^2-x^2}} dy\int_0^y dz$$

$$=48\int_0^{\frac{\sqrt{2}}{2}a} dx\int_0^{x} y\,dy + 48\int_{\frac{\sqrt{2}}{2}a}^{a} dx\int_0^{\sqrt{a^2-x^2}} y\,dy.$$

$$\frac{1}{2}y^2\Big|_0^x$$

$$=48\int_0^{\frac{\sqrt{2}}{2}a}\frac{1}{2}x^2\,dx + \overset{24}{\cancel{48}}\int_{\frac{\sqrt{2}}{2}a}^{a}\cancel{\tfrac{1}{2}}(a^2-x^2)\,dx$$

$$=8(2-\sqrt{2})a^3.$$

7. $D=\{(x,y):y>0\}$ 内，$f(x,y)\in C^1$. 且 $\forall t>0$.
求 $f(tx,ty)=t^{-2}f(x,y)$. 证明：对 $\forall$ 分段光滑的简单闭曲线 L，有 $\oint_L y f(x,y)dx - x f(x,y)dy = 0$

证：$f(tx,ty)=t^{-2}f(x,y)$. 求导(对 t)

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -2\frac{1}{t^3}f(x,y)$$

代入 $t=1$，$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -2f(x,y)$

$$\oint_L y f(x,y)dx - x f(x,y)dy$$

Green公式
$$\iint_\Omega \frac{\partial(-xf(x,y))}{\partial x} - \frac{\partial(y f(x,y))}{\partial y}\,dxdy$$

$$=\iint_\Omega -f(x,y) - x\frac{\partial f}{\partial x} - f(x,y) - y\frac{\partial f}{\partial y}\,dxdy$$

$$=\iint 0\,dxdy = 0$$，得证 $\square$

$$\iint_\Omega \cdots$$

8、设 f、g 在 R 上，$K(x,y)$ 在 R^2 连续 $K>0$，$f,g>0$

$$\int_0^1 f(y)K(x,y)dy = g(x),\quad \int_0^1 g(y)K(x,y)dy = f(x)$$

(1) 证明：若 $m = \min\limits_{0\le x\le 1}\frac{f}{g}$，$M = \max\limits_{0\le x\le 1}\frac{f}{g}$，则 $mM=1$.

(2) 当 $0\le x\le 1$ 时，证：$f=g$.

证明：(1) $g = \int_0^1 f(y)K(x,y)dy = \int_0^1 g(y)\frac{f}{g}K(x,y)dy$

$$\le \int_0^1 M g(y)K(x,y)dy = Mf.$$

即 $M \ge \frac{g}{f}$ 对 $\forall x$ 成立，$\frac{f}{g}\ge\frac{1}{M}$ 对 $\forall x$ 成立

即 $m \ge \frac{1}{M}$，$Mm\ge 1$.

$$f = \int_0^1 g(y)K(x,y)dy = \int_0^1 f(y)\left(\frac{f}{g}\right)^{-1}K(x,y)dy$$

$$\le \frac{1}{m}\int_0^1 f(y)K(x,y)dy = \frac{1}{m}g.$$

$\frac{f}{g}\le\frac{1}{m}$ 对 $\forall x$ 成立，即 $M\le\frac{1}{m}$，$Mm\le 1$.

故 $Mm=1$.

(2) $h=\frac{f}{g}$，
$$\begin{cases}\int_0^1 g(y)h(y)K(x,y) = g(x)\\ \int_0^1 g(y)K(x,y)dy = g(x)h(x)\end{cases}$$

$$g \ge m\int_0^1 g(y)K(x,y).dy$$

则 $Mg(x) \ge \int_0^1 g(y)K(x,y)dy = g(x)h(x).$

当 x 使得 $h(x)$ 最大时，$Mg(x)=g(x)h(x)$

若 $h(y)\neq m$ 则 $g>m\int_0^1 g(y)k(x,y)dy$

即 $Mg(x)>g(x)h(x)$ 矛盾.

故 $h(y)\equiv m$，$M=m$，则 $M=m=1$.

即 $f\equiv g$，证毕. $\square$

附：2021年数分A2第三次小测解答.

证明 $V=\big((2x+y+z)yz,(x+2y+z)zx,(x+y+2z)xy\big)$

是有势场，并求出其所有势函数.

解：$P=(2x+y+z)yz$，$Q=(x+2y+z)zx$，$R=(x+y+2z)xy$

$$\nabla\times V=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)$$

$$\frac{\partial R}{\partial y}=xy+x(x+y+2z),\quad \frac{\partial Q}{\partial z}=zx+x(x+2y+z).$$

则 $\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}=0$，同理，$\frac{\partial P}{\partial z}=\frac{\partial R}{\partial x}$，$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$.

故 V 是无旋场，V 定义域为 R^3，单连通.

故 V 是有势场.

$$\varphi_0(a,b,c)=\int_{(0,0,0)}^{(a,b,c)}\vec F\cdot\vec\tau\,ds$$

(a,b,c)

$$= \int_0^{(a,b,c)} (2x+y+z)\,yz\,dx + (x+2y+z)\,zx\,dy + (x+y+2z)\,xy\,dz$$

$$= \int_0^{a} \left(2t + \frac{b}{a}t + \frac{c}{a}t\right)\frac{bc}{a^2}t^2 + \left(t + \frac{2b}{a}t + \frac{c}{a}t\right)\frac{c}{a}t^2 \times \frac{b}{a} + \left(t + \frac{b}{a}t + \frac{2c}{a}t\right)\frac{b}{a}t^2 \times \frac{c}{a}\,dt$$

$$= \int_0^{a} \frac{bc}{a^2}t^2\left(4t + \frac{4b}{a}t + \frac{4c}{a}t\right)dt$$

$$= \frac{4bc}{a^3}\int_0^{a} t^2(at + bt + ct)\,dt$$

$$= \frac{4bc(a+b+c)}{a^3}\int_0^{a} t^3\,dt$$

$$= abc(a+b+c)$$

则 $\varphi_0(x,y,z) = xyz(x+y+z)$

所有势函数为 $\varphi = xyz(x+y+z) + C \quad (C \in R)$

验证：$\dfrac{\partial \varphi}{\partial x} = yz(x+y+z) + xyz$

$$= 2xyz + yz(y+z)$$

$$= (2x+y+z)yz$$，成立.

即所求正是 v 的势函数.