中国科学技术大学数学科学学院

2021～2022 学年第一学期期末试卷(A卷)

课程名称：微分流形　课程编号：MATH5003P

开课院系：数学科学学院　考试形式：闭卷

姓名：__________　学号：____________　班级：____________

题 号	一	二	三	四	五	六	七	八	九	总 分
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General facts about this exam:

- This is a closed-book examination.
- There are 9 problems (10 pages including this cover) in total.
- Read all the questions carefully before starting to work.
- Attempt to answer all questions for partial credit.
- You have two hours to complete this exam.

Smoothness assumptions in this exam:

- All manifolds in this exam are smooth and have dimension at least 1.
- All maps in this exam are smooth.
- All vector fields in this exam are smooth.
- All distributions in this exam are smooth.
- All group actions in this exam are smooth.
- All tensor fields in this exam are smooth.
- All differential forms in this exam are smooth.

☺ Have A SMOOTH Exam! ☺

Problem 1 (4 points each, 20 points in total)

State the following theorems and briefly explain their importance (or how they are used)

(1) Sard's theorem:

Importance:

(2) The Whitney embedding theorem:

Importance:

(3) Stokes' formula (theorem):

Importance:

(4) Cartan's closed subgroup theorem:

Importance:

(5) The Mayer-Vietories sequence theorem:

Importance:

(6) The Poincaré duality theorem:

Importance:

Problem 2 (2 points each, 20 points in total)

Which of the following statements are correct? Put a "T" before correct ones, and an "F" before wrong ones.

( ) Any topological manifold admits at least one smooth structure.

( ) The restriction of any smooth function on $\mathbb{R}^{n+1}$ to S^n is a smooth function on S^n.

( ) Any smooth function on a smooth manifold has at least one critical point.

( ) Any smooth manifold can be embedded into $\mathbb{RP}^N$ for N large enough.

( ) If $f : M \to N$ is a submersion at p, then there is a neighborhood U of p so that f is a submersion at each $q \in U$.

( ) Any vector field X on M defines an integrable 1-dimensional distribution.

( ) If M is a connected smooth manifold and $S \subset M$ is a smooth submanifold of codimension 2, then the complement $M - S$ is connected.

( ) $\mathrm{GL}(n, \mathbb{R})$ is a connected Lie group of dimension n^2.

( ) Suppose a Lie group G acts smoothly on a smooth manifold M. Then the quotient topology on the orbit space M/G is Hausdorff.

( ) If M is a non-orientable manifold with boundary, then ∂M is non-orientable.

( ) If M is an orientable connected smooth manifold of dimension m, and $\omega \in \Omega_c^m(M)$ satisfies $\int_M \omega = 0$, then ω is exact.

Problem 3 (3 points each, 15 points in total)

For each of the following statements, write down an example.(No detail is needed.)

(1) An embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

(2) Three smooth vector fields on S^3 that are everywhere linearly independent.

(3) A connected Lie group G so that the exponential map $\exp : \mathfrak{g} \to G$ is not surjective.

(4) A smooth map $f : S^{2022} \to S^{2022}$ that is not orientation preserving.

(5) A smooth manifold M so that $\dim M = 2022$ and $\dim H^1_{dR}(M) = +\infty$.

(6) A smooth manifold of dimension 2022 that is not the boundary of any compact smooth manifold (with boundary) of dimension 2023.

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Problem 4 (4 points each, 20 points in total)

Write down the answers directly: (I don't need any detail)

(1) Suppose $X = x\frac{\partial}{\partial x} + y^3\frac{\partial}{\partial z}$, $Y = x^2\frac{\partial}{\partial y} - xz\frac{\partial}{\partial z}$, then

$[X, Y] =$ ____________________________

(2) Suppose $\omega = \sin x dx + e^{2y}dy - ydz$, $\eta = -\sin x dx + ydz$, then

$\omega \wedge \eta =$ ____________________________

(3) Suppose $X = x\frac{\partial}{\partial x} + y^3\frac{\partial}{\partial z}$, $\omega = \sin x dx + e^{2y}dy - ydz$, then

$\mathcal{L}_X\omega =$ ____________________________

(4) Let $M = S^2 \times S^4$, then

$\chi(M) =$ ____________________________

(5) Let $G = SO(7)$, then

$\dim G =$ ____________________________

(6) Consider the map $f : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$f(x, y, s, t) = (x^2 + y, x^2 + y^2 + y + s^2 + t^2),$$

then the set of critical points of f is

$\mathrm{Crit}(f) =$ ____________________________

Problem 5 (15 points)

On $\mathbb{R}^3$, consider vector fields

$$X = z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, \qquad Y = z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}.$$

(1) Write down the definitions of <u>distribution</u> and <u>integrable distribution</u>.

(2) Find the maximal subset $U \subset \mathbb{R}^3$ so that $\mathcal{V}_p := \operatorname{span}\{X_p, Y_p\}$ defines a 2-dimensional distribution.

(3) Prove $\mathcal{V}$ integrable on U, and find the integral manifold through the point (1,2,3).

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Problem 6 (10 points)

Let X be a nowhere vanishing smooth vector field on M. Prove: There exists a smooth 1-form $\omega \in \Omega^1(M)$ so that $\omega(X) = 1$.

Problem 7 (25 points)

Let $f : S \to M$ be a smooth map between two smooth manifolds. For each k, define $\Omega^k(f) = \Omega^k(M) \oplus \Omega^{k-1}(S)$, and define three maps $\widetilde{d}, \alpha, \beta$ as follows:

$$\widetilde{d} : \Omega^k(f) \to \Omega^{k+1}(f), \qquad (\omega, \theta) \mapsto (d\omega, f^*\omega - d\theta),$$
$$\alpha : \Omega^{k-1}(S) \to \Omega^k(f), \qquad \theta \mapsto (0, \theta),$$
$$\beta : \Omega^k(f) \to \Omega^k(M), \qquad (\omega, \theta) \mapsto \omega.$$

(1) Prove: $\widetilde{d} \circ \widetilde{d} = 0,\ \alpha \circ d = -\widetilde{d} \circ \alpha,\ \ \beta \circ \widetilde{d} = d \circ \beta$.

(2) Define
$$H^k(f) = \frac{\ker(\widetilde{d} : \Omega^k(f) \to \Omega^{k+1}(f))}{\mathrm{Image}(\widetilde{d} : \Omega^{k-1}(f) \to \Omega^k(f))}.$$
Then the maps α, β induce maps
$$\alpha^* : H^{k-1}_{dR}(S) \to H^k(f), \qquad [\theta] \mapsto [\alpha(\theta)]$$
$$\beta^* : H^k(f) \to H^k_{dR}(M), \qquad [(\omega, \theta)] \mapsto [\beta(\omega, \theta)].$$
Recall that the map f also induces a pull-back map $f^* : H^k_{dR}(M) \to H^k_{dR}(S)$. Prove: $\beta^* \circ \alpha^* = 0$, $f^* \circ \beta^* = 0$, $\alpha^* \circ f^* = 0$,

(3) Prove: $\mathrm{Image}(\beta^*) = \ker(f^*)$, $\mathrm{Image}(f^*) = \ker(\alpha^*)$, $\mathrm{Image}(\alpha^*) = \ker(\beta^*)$.

(4) Prove: if $f, g : S \to M$ are homotopic smooth maps, then $H^k(f) \simeq H^k(g)$.

Problem 8 (10 points)

Suppose $m = k + l$, where $k, l \geq 1$. Prove: There do not exist k-dimensional manifold M and l-dimensional manifold N so that $M \times N$ is diffeomorphic to S^m.

Problem 9 (15 points)

A planar polygon (oriented) with edge lengths $l_1, \cdots, l_n > 0$ is represented by an ordered n-tuples of points $p_1, \cdots, p_n \in \mathbb{R}^2$ such that for each $1 \le i \le n$, the distance $\|p_{i+1} - p_i\|$ between p_i and p_{i+1} is l_i (where we use the convention that $p_{n+1} = p_1$). We will regard two such polygons as the same if one can be obtained from the other after rotation and translation. Denote the origin by O, and denote the positive x-axis by X, namely

$$X = \{(x, 0) \mid x > 0\} \subset \mathbb{R}^2.$$

In what follows, assume $n = 4$.

(a) For any $(l_1, \cdots, l_n) \in (\mathbb{R}_{>0})^n$, we consider the set
$\mathcal{P}(l_1, \cdots, l_n) := \{(p_1, \cdots, p_n) : \|p_{i+1} - p_i\| = l_i \ (1 \le i \le n), p_1 = O, p_n \in X\} \subset (\mathbb{R}^2)^n$.
Explain why $\mathcal{P}(l_1, \cdots, l_n)$ can be viewed as the space of planar polygons with edge lengths $l_1, \cdots, l_n$ up to rotation and translation.

(b) To study $\mathcal{P}(l_1, \cdots, l_n)$, we introduce an auxiliary space (for $l_1, \cdots, l_{n-1} > 0$)
$\mathcal{A}(l_1, \cdots, l_{n-1}) := \{(p_1, \cdots, p_n) : \|p_{i+1} - p_i\| = l_i (1 \le i \le n-1), p_1 = O, p_n \in X\} \subset (\mathbb{R}^2)^n$.
Prove: $\mathcal{A}(l_1, \cdots, l_{n-1})$ is a smooth manifold of dimension $n - 2$.

(c) Prove that there exists a dense subset of $(l_1, \cdots, l_n)$ in $(\mathbb{R}_{>0})^n$ such that $\mathcal{P}(l_1, \cdots, l_n)$ is either empty, or a smooth manifold of dimension $n - 3$.