Stochastic Processes, MA04243, Spring 2019, Final

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1. Let p be a transition probability on a countable state space S. For the Markov chain $X = (X_n)_{n\geq 0}$ with transition probability p, and $A \subset S$, define

$$V_A = \inf\{n \geq 0 : X_n \in A\}.$$

Also define $f(x) = E_x V_A$ for $x \in S$. Clearly $f(x) = 0$ for every $x \in A$. Now, first show that

$$f(x) = 1 + \sum_y p(x,y) f(y), \quad x \in A^c;$$

then for any bounded function g on S satisfying

$$g(x) = 1 + \sum_y p(x,y) g(y), \quad x \in A^c,$$

show that $g(X_{n\wedge V_A}) + n \wedge V_A$ is a martingale.

2. Let $(B_t)_{t\geq 0}$ be a real Brownian motion started from 0 and for every $a \geq 0$ set

$$T_a = \inf\{t \geq 0 : B_t = a\}.$$

Show that the process $(T_a)_{a\geq 0}$ has stationary independent increments.

3. Let $(B_t)_{t\geq 0}$ be a real Brownian motion started from 0 and for every $a \geq 0$ set

$$T_a = \inf\{t \geq 0 : B_t = a\}.$$

Show that the process $(T_a)_{a\geq 0}$ has the following scaling property:
For every $\lambda > 0, (T_{\lambda a})_{a\geq 0} \overset{d}{=} (\lambda^2 T_a)_{a\geq 0}$.

4. Let $B_t = (B_t^1, B_t^2, \cdots, B_t^d)$ be a d-dimensional Brownian motion started from 0. Use $|x|$ to denote the Euclidean distance between the two points x and 0 in $\mathbb{R}^d$ and for every $a > 0$ set

$$U_a = \inf\{t \geq 0 : |B_t| = a\}.$$

Compute $E[U_a]$.

5. Let $(B_t)_{t\geq 0}$ be an $(\mathscr{F}_t)$-Brownian motion started from 0, and for $a > 0$ set

$$\sigma_a = \inf\{t \geq 0 : B_t \leq t - a\}.$$

Assume that you have already shown that σ_a is a stopping time and that $\sigma < \infty$ a.s. Now using an appropriate exponential martingale, show that, for every $\lambda \geq 0$,

$$E[\exp(-\lambda \sigma_a)] = \exp(-a(\sqrt{1+2\lambda} - 1)).$$