2017-2018年数学分析A2期末解答.

1、讨论函数 $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ 在(0,0)处的连续性和可微性.

解：当 $(x,y)\neq(0,0)$ 时，$\sqrt{x^2+y^2}\geqslant\sqrt{2|xy|}$

则 $|f(x,y)-0|\leq\dfrac{|xy|}{\sqrt{2|xy|}}=\dfrac{\sqrt{2}}{2}\sqrt{|xy|}\leq\sqrt{|xy|}$

则对 $\forall\varepsilon>0$，存在 $\delta=\sqrt{\varepsilon}$，当 $x,y\in B_\delta(0,0)$ 时，

有 $|f(x,y)|<\varepsilon$，即 $f(x,y)\in B_\varepsilon(0)$.

故 f 在(0,0)处连续.

f在(0,0)处不可微，理由如下：

若f在(0,0)处可微，则f在(0,0)处任意方向的方向导数存在，则设一个方向为 $u=(\cos a,\sin a)$.

则 $\dfrac{\partial f}{\partial u}(0,0)=\lim\limits_{t\to0}\dfrac{1}{t}[f(\vec{0}+t\vec{u})-f(0,0)]=\cos a\sin a.$

而 $\dfrac{\partial f}{\partial x}(0,0)=\lim\limits_{t\to0}\dfrac{f(t,0)-f(0,0)}{t}=0.$

$\dfrac{\partial f}{\partial y}(0,0)=\lim\limits_{t\to0}\dfrac{f(0,t)-f(0,0)}{t}=0$

由 $\dfrac{\partial f}{\partial u}=\dfrac{\partial f}{\partial x}u_1+\dfrac{\partial f}{\partial y}u_2$ 知 $\dfrac{\partial f}{\partial u}=0$.

而当 $a\neq k\pi$ 或 $(k+\frac{1}{2})\pi$ 时 $\dfrac{\partial f}{\partial u}\neq0$，矛盾

故f在(0,0)处不可微

2、在椭球面 $2x^2+2y^2+z^2=1$ 上求一点，使函数 $f(x,y,z)=x^2+y^2+z^2$ 在沿该点方向 $l=\frac{\sqrt{2}}{2}i-\frac{\sqrt{2}}{2}j$ 的方向导数值最大.

解：$i=(1,0,0)$，$j=(0,1,0)$，$l=\frac{\sqrt{2}}{2}(i-j)=(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0)$

由 $\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta+\frac{\partial f}{\partial z}\cos\gamma$

当 $(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})$ 与 $(\cos\alpha,\cos\beta,\cos\gamma)$ 平行时，

$|\frac{\partial f}{\partial u}|$ 取到最大值，即 $\frac{\partial f}{\partial u}$ 最大或最小

$(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})=(2x,2y,2z)$

$(2x,2y,2z)$ 与 $(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0)$ 平行，则 $z=0$，$x+y=0$

联立 $\begin{cases}2x^2+2y^2+z^2=1\\ z=0\\ x+y=0\end{cases}$ 得 $\begin{cases}x=\frac{1}{2}\\ y=-\frac{1}{2}\\ z=0\end{cases}$ 或 $\begin{cases}x=-\frac{1}{2}\\ y=\frac{1}{2}\\ z=0\end{cases}$

$$\frac{\partial f}{\partial l}(\tfrac{1}{2},-\tfrac{1}{2},0)=\frac{\sqrt{2}}{2}\frac{\partial f}{\partial x}(\tfrac{1}{2},-\tfrac{1}{2},0)-\frac{\sqrt{2}}{2}\frac{\partial f}{\partial y}(\tfrac{1}{2},-\tfrac{1}{2},0)$$
$$=\frac{\sqrt{2}}{2}\times2\times\frac{1}{2}-\frac{\sqrt{2}}{2}\times2\times(-\frac{1}{2})$$
$$=\sqrt{2}.$$

则 $\frac{\partial f}{\partial l}(-\frac{1}{2},\frac{1}{2},0)=-\frac{\partial f}{\partial l}(\frac{1}{2},-\frac{1}{2},0)=-\sqrt{2}$.

故 $(\frac{1}{2},-\frac{1}{2},0)$ 点处 f 沿方向 l 的方向导数最大，为 $\sqrt{2}$

3、设变换 $\begin{cases}u=x+a\sqrt{y}\\ v=x+2\sqrt{y}\end{cases}$ 把方程 $\frac{\partial^2 z}{\partial x^2}-y\frac{\partial^2 z}{\partial y^2}-\frac{1}{2}\frac{\partial z}{\partial y}=0$

化为 $\frac{\partial^2 z}{\partial u\partial v}=0$，试确定 a 的值

解：$z=z(u,v)=z(x,y)$

则 $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial v}{\partial x}+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}$$
$$=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 u}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}.$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{a}{2}\frac{\partial z}{\partial u}\frac{1}{\sqrt{y}}+\frac{\partial z}{\partial v}\frac{1}{\sqrt{y}}.$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{a}{2}\left(\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial v}{\partial y}\right)\frac{1}{\sqrt{y}}+\frac{a}{2}\left(\frac{\partial z}{\partial u}\right)\left(\frac{1}{\sqrt{y}}\right)'$$

$$+\left(\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial y}\right)\frac{1}{\sqrt{y}}+\left(\frac{\partial z}{\partial v}\right)\left(\frac{1}{\sqrt{y}}\right)'.$$

$$=\frac{a}{2}\left(\frac{\partial^2 z}{\partial u^2}\times\frac{a}{2}\frac{1}{\sqrt{y}}+\frac{\partial^2 z}{\partial u\partial v}\frac{1}{\sqrt{y}}\right)\frac{1}{\sqrt{y}}-\frac{a}{2}\times\frac{1}{2}\frac{\partial z}{\partial u}\frac{1}{y\sqrt{y}}$$

$$+\left(\frac{\partial^2 z}{\partial u\partial v}\times\frac{a}{2}\frac{1}{\sqrt{y}}+\frac{\partial^2 z}{\partial v^2}\frac{1}{\sqrt{y}}\right)\frac{1}{\sqrt{y}}-\frac{1}{2}\frac{\partial z}{\partial v}\frac{1}{y\sqrt{y}}$$

$$=\frac{a^2}{4}\frac{\partial^2 z}{\partial u^2}\frac{1}{y}+\frac{a}{2}\frac{\partial^2 z}{\partial u\partial v}\frac{1}{y}-\frac{a}{4}\frac{\partial z}{\partial u}\frac{1}{y\sqrt{y}}+\frac{a}{2}\frac{\partial^2 z}{\partial u\partial v}\frac{1}{y}$$

$$+\frac{\partial^2 z}{\partial v^2}\frac{1}{y}-\frac{1}{2}\frac{\partial z}{\partial v}\frac{1}{y\sqrt{y}}$$

代入：$\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}-\frac{a^2}{4}\frac{\partial^2 z}{\partial u^2}-\frac{a}{2}\frac{\partial^2 z}{\partial u\partial v}+\frac{a}{4}\frac{\partial z}{\partial u}\frac{1}{\sqrt{y}}$

$$-\frac{a}{2}\frac{\partial^2 z}{\partial u\partial v}-\frac{\partial^2 z}{\partial v^2}+\frac{1}{2}\frac{\partial z}{\partial v}\frac{1}{\sqrt{y}}$$

$$=\frac{a}{4}\frac{\partial z}{\partial u}\frac{1}{\sqrt{y}}+\frac{1}{2}\frac{\partial z}{\partial v}\frac{1}{\sqrt{y}}.$$

即 $\left(1-\frac{a^2}{4}\right)\frac{\partial^2 z}{\partial u^2}-(2-a)\frac{\partial^2 z}{\partial u\partial v}=0$

则 $1-\frac{a^2}{4}=0$，且 $a\neq 2$，解得 $a=-2$.

4. f, g 为 R 上的连续可微函数，$f(0)=g(0)=1$

第二型曲线积分 $\int_A^B y f(x)\,dx+(f(x)+zg(y))\,dy+g(y)\,dz$ 与

$A\to B$ 的路径无关，求出向量场

$\vec{F}(x,y,z)=(yf(x),\ f(x)+zg(y),\ g(y))$ 的势函数.

解：从第二型曲线积分与路径无关可知 F 是保守场

由 F 定义在 R^3 上，R^3 是单连通曲面.

则 F 还是有势场，无旋场.

$P=yf(x)$，$Q=f(x)+zg(y)$，$R=g(y)$

$\frac{\partial R}{\partial y}=\frac{\partial Q}{\partial z}$，即 $g(y)=g'(y)$，且 $g(0)=1$

构造 $G(y)=e^{-y}g(y)$，$G'(y)=e^{-y}(g'(y)-g(y))=0$.

则 $G(y)=e^{-y}g(y)=C$

$g(y)=ce^{y}$，代入 $g(0)=1$ 得 $c=1$.

即 $g(y)=e^{y}$

且 $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$，即 $f(x)=f'(x)$，$f(0)=1$.

同理可得 $f(x)=e^{x}$.

则 $P=ye^{x}$，$Q=e^{x}+ze^{y}$，$R=e^{y}$

$\vec{F}=(P,Q,R)$ 的一个势函数为

$$\varphi_0(a,b,c)=\int_{(0,0,0)}^{(a,b,c)} ye^{x}dx+(e^{x}+ze^{y})dy+e^{y}dz$$

$$=\int_0^a \frac{b}{a}te^{t}+(e^{t}+\frac{c}{a}te^{\frac{b}{a}t})\times\frac{b}{a}+e^{\frac{b}{a}t}\times\frac{c}{a}\,dt.$$

$$=\int_0^a \frac{b}{a}(t+1)e^{t}+\frac{c}{a}(\frac{b}{a}t+1)e^{\frac{b}{a}t}\,dt.$$

$$=\frac{b}{a}te^{t}\Big|_0^a+\frac{c}{b}te^{t}\Big|_0^b$$

$$=be^{a}+ce^{b}$$

则所有的势函数为

$\varphi(x,y,z)=ye^{x}+ze^{y}+C$，C为常数.

5. 计算第二型曲线积分 $\int_{L^+}\frac{(x-y)dx+(x+4y)dy}{x^2+4y^2}$，其中L为不通过原点的简单光滑闭曲线，L+为逆时针方向.

解：① 若L围成的区域不含原点，$P=\frac{x-y}{x^2+4y^2}$，$Q=\frac{x+4y}{x^2+4y^2}$.

由Green公式：

$$\int_{L^+}\frac{(x-y)dx+(x+4y)dy}{x^2+4y^2}=\iint_{\Sigma}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

$$\frac{\partial Q}{\partial x}=\frac{x^2+4y^2-(x+4y)2x}{(x^2+4y^2)^2}=\frac{4y^2-8xy-x^2}{(x^2+4y^2)^2},$$

$$\frac{\partial P}{\partial y}=\frac{-(x^2+4y^2)-(x-y)8y}{(x^2+4y^2)^2}=\frac{4y^2-8xy-x^2}{(x^2+4y^2)^2}.$$

则该第二型曲线积分为0.

② 若L围成的区域包含原点，则取ε足够小，使得

$B:\{(x,y)\mid x^2+4y^2=\varepsilon^2\}\subseteq\Omega$（Ω为L围成的区域）.

则 $\int_{L^+}Pdx+Qdy=\int_{B^+}Pdx+Qdy=\int_{B^+}\frac{1}{\varepsilon^2}[(x-y)dx+(x+4y)dy]$

$=\frac{1}{\varepsilon^2}\int_{B^+}(x-y)dx+(x+4y)dy.$

曲线B的参数形式为 $\begin{cases}x=\varepsilon\cos\theta\\ y=\frac{\varepsilon}{2}\sin\theta\end{cases}$ $\theta\in[0,2\pi]$.

原式 $=\frac{1}{\varepsilon^2}\int_0^{2\pi}(\varepsilon\cos\theta-\frac{\varepsilon}{2}\sin\theta)(-\varepsilon\sin\theta)+(\varepsilon\cos\theta+2\varepsilon\sin\theta)(\frac{\varepsilon}{2}\cos\theta)d\theta$

$=\int_0^{2\pi}(\cos\theta-\frac{1}{2}\sin\theta)(-\sin\theta)+(\cos\theta+2\sin\theta)(\frac{1}{2}\cos\theta)d\theta$

$=\int_0^{2\pi}\frac{1}{2}\sin^2\theta-\sin\theta\cos\theta+\frac{1}{2}\cos^2\theta+\sin\theta\cos\theta\,d\theta$

$=\int_0^{2\pi}\frac{1}{2}d\theta=\pi.$

6、计算第二型曲面积分 $\iint_{\Sigma} xdydz+ydzdx+zdxdy$.

其中 Σ 为圆柱面 $x^2+y^2=1$ 被 $z=0$ 和 $z=3$ 所截部分的外侧

解：记面 $\begin{cases} x^2+y^2\le 1 \\ z=0 \end{cases}$ 和 $\begin{cases} x^2+y^2\le 1 \\ z=3 \end{cases}$ 分别为 Σ_1, Σ_2，外侧

则 $\iint_{\Sigma+\Sigma_1+\Sigma_2} xdydz+ydzdx+zdxdy$

$=\iiint_{\Omega} 3\,dxdydz = 3\times\pi\times 3 = 9\pi$.

且 $\iint_{\Sigma_1} xdydz+ydzdx+zdxdy = 0$.

$\iint_{\Sigma_2} xdydz+ydzdx+zdxdy = 3\pi$.

则 $\iint_{\Sigma} xdydz+ydzdx+zdxdy = 6\pi$.

7、设函数 $f(x,y)$ 在原点附近有直到二阶的各种形式的连续偏导数，S是球面 $x^2+y^2+z^2=r^2$ $(r>0)$，

求 $\lim\limits_{r\to 0^+}\dfrac{\iint_S [f(x,y)-f(0,0)]d\sigma}{r^4}$ 的值.

解：由Taylor展开公式，

$f(x,y)-f(0,0) = x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+\frac{1}{2}x^2\frac{\partial^2 f}{\partial x^2}+xy\frac{\partial^2 f}{\partial x\partial y}+\frac{1}{2}y^2\frac{\partial^2 f}{\partial y^2}+$余项

余项记为 $\beta(x,y)$，其是 x^2+y^2，即 r^2 的高阶无穷小.

则 $\lim\limits_{r\to 0^+}\dfrac{\iint_S [f(x,y)-f(0,0)]d\sigma}{r^4}$

$=\lim\limits_{r\to 0}\dfrac{\iint_S [x\frac{\partial f}{\partial x}(0,0)+y\frac{\partial f}{\partial y}(0,0)+\frac{1}{2}x^2\frac{\partial^2 f}{\partial x^2}(0,0)+\frac{1}{2}y^2\frac{\partial^2 f}{\partial y^2}(0,0)+xy\frac{\partial^2 f}{\partial x\partial y}(0,0)+o(r^2)]d\sigma}{r^4}$

$=\lim\limits_{r\to 0}\dfrac{\iint_S [x\frac{\partial f}{\partial x}(0,0)+y\frac{\partial f}{\partial y}(0,0)+\frac{1}{2}x^2\frac{\partial^2 f}{\partial x^2}(0,0)+\frac{1}{2}y^2\frac{\partial^2 f}{\partial y^2}(0,0)+xy\frac{\partial^2 f}{\partial x\partial y}]d\sigma+o(r^4)}{r^4}$

$=\lim\limits_{r\to 0}\dfrac{\iint_S [x\frac{\partial f}{\partial x}(0,0)+y\frac{\partial f}{\partial y}(0,0)+\frac{1}{2}x^2\frac{\partial^2 f}{\partial x^2}(0,0)+\frac{1}{2}y^2\frac{\partial^2 f}{\partial y^2}(0,0)+xy\frac{\partial^2 f}{\partial x\partial y}(0,0)]d\sigma}{r^4}$

记 $a=\frac{\partial f}{\partial x}(0,0)$，$b=\frac{\partial f}{\partial y}(0,0)$，$c=\frac{\partial^2 f}{\partial x^2}(0,0)$，$d=\frac{\partial^2 f}{\partial y^2}(0,0)$，$e=\frac{\partial^2 f}{\partial x\partial y}(0,0)$

$$原式=\lim_{r\to 0}\frac{\iint_S (ax+by+\frac{c}{2}x^2+\frac{d}{2}y^2+exy)d\sigma}{r^4}$$

$$=\lim_{r\to 0}\frac{\iint_S (\frac{c}{2}x^2+\frac{d}{2}y^2)d\sigma}{r^4}$$

参数形式：$\begin{cases} x=r\sin\theta\cos\varphi \\ y=r\sin\theta\sin\varphi \\ z=r\cos\theta \end{cases}$ $\theta\in[0,\pi]$，$\varphi\in[0,2\pi]$.

$\|r_\theta\times r_\varphi\|=r^2\sin\theta$.

则 $\iint_S (\frac{c}{2}x^2+\frac{d}{2}y^2)d\sigma$

$$=\iint_\Delta (\frac{c}{2}r^2\sin^2\theta\cos^2\varphi+\frac{d}{2}r^2\sin^2\theta\sin^2\varphi)r^2\sin\theta\, d\theta d\varphi.$$

$$原极限=\iint_\Delta (\frac{c}{2}\sin^2\theta\cos^2\varphi+\frac{d}{2}\sin^2\theta\sin^2\varphi)\sin\theta\, d\theta d\varphi.$$

$$=\frac{c}{2}\iint_\Delta \sin^3\theta\cos^2\varphi\, d\theta d\varphi+\frac{d}{2}\iint_\Delta \sin^3\theta\sin^2\varphi\, d\theta d\varphi.$$

$$=\frac{c}{2}\times 2\times\frac{2!!}{3!!}\times\pi+\frac{d}{2}\times 2\times\frac{2!!}{3!!}\times\pi$$

$$=\frac{2}{3}\pi(c+d)$$

$$=\frac{2}{3}\pi\left(\frac{\partial^2 f}{\partial x^2}(0,0)+\frac{\partial^2 f}{\partial y^2}(0,0)\right).$$

8、设 $f(x,y)$ 为具有二阶连续偏导数的二次齐次函数.

即 $\forall x,y,t$，成立 $f(tx,ty)=t^2f(x,y)$.

(1) 证明：$xf'_x(x,y)+yf'_y(x,y)=2f(x,y)$.

(2) 设 D 是由 L：$x^2+y^2=4$ 所围成的区域，证明：

$$\int_L f(x,y)ds=\iint_D\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)dxdy.$$

解：(1) 证明：$f(tx,ty)=t^2f(x,y)$

则对 t 求导，$x\frac{\partial f}{\partial x}(tx,ty)+y\frac{\partial f}{\partial y}(tx,ty)=2tf(x,y)$.

代入 $t=1$，得 $xf_x'(x,y)+yf_y'(x,y)=2f(x,y)$. $\square$

(2) 证明：$\int_L f(x,y)ds=\int_L\left(\frac{1}{2}x\frac{\partial f}{\partial x}+\frac{1}{2}y\frac{\partial f}{\partial y}\right)ds$.

$$=\int_L\left(-\frac{\partial f}{\partial y},\frac{\partial f}{\partial x}\right)\cdot\left(-\frac{1}{2}y,\frac{1}{2}x\right)d\sigma$$

且 x,y 满足 $x^2+y^2=4$，则 $\left(\frac{1}{2}x,\frac{1}{2}y\right)$ 为单位法向量

则 $\left(-\frac{1}{2}y,\frac{1}{2}x\right)$ 为单位切向量，且绕 L 逆时针旋转.

则 $\int_L f(x,y)ds=\int_L\left(-\frac{\partial f}{\partial y},\frac{\partial f}{\partial x}\right)\cdot t\,d\sigma$

$$=\iint_D\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)dxdy.$$

其中最后一个等号是由 Green 公式得出 $\square$