2018年秋季学期 微分流形期末考试
2018 Fall Final Exam: Differential Manifolds

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Problem 1 (20 points, 4 points each)

Write down the definitions of the following conceptions.

(1) A topological manifold is a topological space M such that ...

(2) A partition of unity subordinate to a locally finite covering $\{U_i\}$ of M is a family of smooth functions $\{\rho_i\}$ such that ...

(3) An immersion $f : M \to N$ is a smooth map such that ...

(4) A Lie group is a smooth manifold G such that ...

(5) An integral curve of a smooth vector field X is a map $\gamma : I \to M$ such that ...

(6) A vector bundle of rank r is a triple $\{\pi, E, M\}$ where $\pi : E \to M$ is a smooth surjective map such that ...

Problem 2 (20 points, 2 points each)

TRUE or FALSE.

( ) If $f : M \to N$ is a smooth map, and Z is a smooth submanifold of N, then $f^{-1}(Z)$ is a smooth submanifold of M;

( ) $\mathbb{T}^2$ can be embedded into $\mathbb{R}^3$;

( ) There exists a smooth vector field X on S^{2n+1} so that $X_p \neq 0, \ \ \forall p \in S^{2n+1}$;

( ) The exponential map $\exp : \mathfrak{g} \to G$ is always surjective;

( ) Any compact smooth manifold is orientable;

( ) For any smooth vector field X and any smooth k-form ω, one has $d\mathcal{L}_X\omega = \mathcal{L}_X d\omega$;

( ) If $w \in B^k(M)$ is compactly supported, then $w \in B_c^k(M)$;

( ) S^2 is a Lie Group;

( ) For any integer k, one can find a smooth map $f : S^1 \to S^1$ whose degree equals k;

( ) The Euler characteristic of $\mathbb{T}^5 \times S^6$ is 0;

( ) If M, N are compact, then $H_{dR}^k(M \times N) = H_{dR}^k(M) \times H_{dR}^k(N)$.

Problem 3 (15 points, 3 points each) Write down the inclusion relation:

(1)_____⊂_____⊂_____

A={ smooth manifolds }, B={ Lie Groups }, C={ orientable manifolds };

(2)_____⊂_____⊂_____

A={ tensor fields on M }, B={ differential forms on M }, C={ volume forms on M };

(3)_____⊂_____⊂_____

A={ $f : M \to N$ is a submersion }, B={ $f : M \to N$ is a smooth map }, C={ $f : M \to N$ is a local diffeomorphism };

(4)_____⊂_____⊂_____

A={ vector bundles }, B={ distributions }, C={ tangent bundles };

(5)_____⊂_____⊂_____

A={ manifolds with finite Betti numbers }, B={ compact manifolds }, C={ manifolds with finite good cover };

(6)_____⊂_____⊂_____

A={ $f : M \to N$ is a diffeomorphism }, B={ $f : M \to N$ is a homeoporphism }, C={ $f : M \to N$ is a homotopy equivalence }.

Problem 4 (20 points, 4 points each)

Let $M = \mathbb{R}^4$ with coordinate $\{x, y, z, w\}$. Let

$$X = w\partial_x + y\partial_z, \quad Y = x^2\partial_y - \partial_z + \sin x\partial_w$$

be vector fields on M. Let

$$\omega = 2xdx + e^y dz - ydw, \quad \eta = dy \wedge dz - dx \wedge dw$$

be differential forms on M. Let

$$\phi : \mathbb{R}^2 \to \mathbb{R}^4, \ \ (t, s) \mapsto (x, y, z, w) = (t^2, s, \cos t \sin t),$$

$$\psi : \mathbb{R}^4 \to \mathbb{R}^2, \ \ (x, y, z, w) \mapsto (x, y, z, w) = (u, v) = (x + y^2, z - e^w)$$

be smooth maps. Compute the following:

(1)$[X, Y] =$ _____

(2)$d\omega =$ _____

(3)$\omega \wedge \eta =$ _____

(4)$\iota_X \eta =$ _____

(5)$\phi^*\omega =$ _____

(6)$d\psi_{(1,1,0,0)}(Y) =$ _____.

Problem 5 (15 points)

Let M be a smooth manifold, and $\theta \in \Omega^1(M)$ be an exact 1-form.

(1) $\forall k$, define $d_\theta : \Omega^k(M) \to \Omega^{k+1}(M)$ by $d_\theta(\omega) = d\omega + \theta \wedge \omega$. Prove that

$$d_\theta(d_\theta(\omega)) = 0 \ \ \forall \omega \in \Omega^k(M).$$

(2) Consider the complex

$$0 \to \Omega^0(M) \xrightarrow{d_\theta} \Omega^1(M) \xrightarrow{d_\theta} \cdots \xrightarrow{d_\theta} \Omega^{n-1}(M) \xrightarrow{d_\theta} \Omega^n(M) \to 0.$$

Please define $Z^k_\theta(M)$, $B^k_\theta(M)$, $H^k_\theta(M)$.

(3) Prove that $H^k_\theta(M)$ is isomorphic to $H^k_{dR}(M)$.

Problem 6 (10 points)

Consider $M = \mathbb{R}^3$ with standard coordinates $\{x, y, z\}$. Consider the distribution $\mathcal{V} = Ker(ydx - dz)$.

(1) Find vector fields X, Y so that $\mathcal{V}_p = Span(X_p, Y_p)$ for all p;

(2) Is $\mathcal{V}$ integrable? Justify your answer.

Problem 7 (15 points)

Let

$$X = S^4 = \{(x, y, z, u, v) \in \mathbb{R}^5 : x^2 + y^2 + z^2 + u^2 + v^2 = 1\}$$

be the standard 4-sphere in $\mathbb{R}^5$, and let

$$Y = \{(x, y, z, u, v) \in \mathbb{R}^5 - \{0\} : x^2 + y^2 + z^2 = u^2 + v^2\}$$

.

(1) Prove Y is a smooth submanifold of $\mathbb{R}^5$;

(2) Prove that X, Y intersects transversally;

(3) Let $M = X \cap Y$. Compute all the de Rham cohomology groups of M.

Problem 8 (15 points)

Let G be a compact Lie group.

(1) Let $Inv : G \to G$ be the inversion $g \mapsto g^{-1}$. Prove that one has $(Inv^*\omega)_e = (-1)^k \omega_e \ \ \forall \omega \in \Omega^k(G)$; (Hint: first prove $k = 1$.)

(2) Define the "left-invariant k- form" on G: Let $\omega \in \Omega^k(G)$ be a smooth k-form on G, we say ω is left-invariant if ...

(3) Prove that $d\omega = 0$ if $\omega \in \Omega^k(G)$ is both left/right-invariant.

Problem 9 (20 points)

Let M be the extended complex plane $M = \mathbb{C} \cup \{\infty\}$.

(1) How to identify M with S^2? (So M is a compact, connected, orientable manifold.)

(2) Let $a_1, \cdots, a_n \in \mathbb{C}$. Consider the map $F : M \to M$ defined by

$$F(z) = \begin{cases} z^n + a_1 z^{n-1} + \cdots, & z \in \mathbb{C} \\ \infty, & z = \infty. \end{cases}$$

Prove that F is a smooth map.

(3) Prove F is homotopic to

$$F_0(z) = \begin{cases} z^n, & z \in \mathbb{C} \\ \infty, & z = \infty. \end{cases}$$

(4) Find the degree of F.

(5) Expalin why your results above imply the fundamental theorem of algebra: Any polynomial equation has at least one complex solution.