Functional Analysis (H) Midterm USTC-2015F

Your name: **Solutions**

1.(20') • **You only need to answer 4 out of the 5 parts for this problem.**

• Check the four problems you want to be graded.

Write down the **definitions** of

☐ We call a function $p : X \to \mathbb{R}$ on a topological vector space X a seminorm if ...
for any $x, y \in X$ and all scalars α, one has

$$p(x+y) \leq p(x) + p(y) \quad \text{and} \quad p(\alpha x) = |\alpha| p(x).$$

☐ We say a subset E in a topological vector space X is bounded if ...
for any neighborhood U of 0 in X, there exists $s > 0$ such that $E \subset tU$ for all $t > s$.

☐ A family $\Lambda = \{L_\alpha\}$ of continuous linear operators from a topological vector space X to a topological vector space Y is said to be equicontinuous if ...
for any neighborhood V of 0 in Y, one can find a neighborhood U of 0 in X so that $L_\alpha(U) \subset V$ for any $L_\alpha \in \Lambda$.

☐ The weak $*$ topology on the dual space X^* of a topological vector space X is defined to be ...
the weakest topology on X^* so that for any $x \in X$, the map $ev_x : X^* \to \mathbb{F}, L \mapsto L(x)$ is continuous.

☐ The annihilator of a vector subspace M in a Banach space X is defined to be the set ...

$$M^\perp = \{x^* \in X^* \mid \langle x, x^* \rangle = 0, \forall x \in M\}.$$

2.(20') • **You only need to answer 4 out of the 5 parts for this problem**.

• Check the four problems you want to be graded.

Write down the statement of the following theorems

☐ The Hahn-Banach theorem: Let X be a real vector space, $p : X \to \mathbb{R}$ a quasi-seminorm. Suppose $Y \subset X$ is a subspace, and $l : Y \to \mathbb{R}$ is a linear functional on Y that is dominated by p. Then there exists a linear functional $L : X \to \mathbb{R}$ on X that extends l and is dominated by p.

☐ The closed graph theorem: Let X, Y be F-spaces, and $L : X \to Y$ be a linear operator whose graph Γ_L is closed in $X \times Y$. Then L is continuous.

☐ The uniform boundedness principle: Let X be a Banach space, Y a normed vector space, and Λ a family of continuous linear operators from X to Y. Suppose the family Λ is pointwise bounded, i.e. for any $x \in X$, $\sup_{L\in\Lambda} \|Lx\| < \infty$. Then there exists a constant C so that $\|L\| < C$ for any $L \in \Lambda$.

☐ The Banach-Alaoglu theorem: Let X be a topological vector space, and $V \subset X$ a neighborhood of 0 in X. Then the polar set $K = \{L \in X^* \mid |Lx| \leq 1, \forall x \in V\}$ is compact in the weak-* topology.

☐ The Riesz representation theorem (for Hilbert space): Let H be a Hilbert space. Then for any $L \in H^*$, there exists a unique $y \in H$ so that $Lx = (x, y)$ for any $x \in H$.

3.(20') • **You only need to answer 4 out of the 5 parts for this problem.**

• Check the four problems you want to be graded.

For each of the following statements, write down an example (**No detail is needed!**)

☐ A normed vector space which is not a Banach space.
The set of all polynomials with the supreme norm
or $C([0,1])$ with the L^1-norm etc

☐ A linear functional that is not continuous.
Let $X = C([0,1])$ equipped with the L^1-norm, and let $L(f) = f(\frac{1}{2})$.

☐ A topological vector space that is not locally convex.
$L^p([0,1])$ $(0 < p < 1)$, with metric $d(f,g) = \int_0^1 |f(x) - g(x)|^p dx$.

☐ A Banach space whose closed unit ball contains no extreme points.
$c_0 = \{x = (a_1, a_2, a_3, \dots) \in l^\infty \mid \lim_{n\to\infty} a_n = 0\} \subset l^\infty$, equipped with the l^∞ norm.

☐ A separable Banach space whose dual space is not separable.
$l^1 = \{x = (a_1, a_2, a_3, \dots) \mid \sum_n a_n < \infty\}$, equipped with the standard l^1 norm.

[• **<u>You only need to answer 3 out of the following 4 problems!</u>**]

[• 20 points each. Check the three problems you want to be graded!]

Answer the following problems

☐ 4. For any $f \in L^2(\mathbb{R})$ we define Lf be the function

$$(Lf)(x) = f(|x|).$$

(a) Prove: $L \in \mathcal{L}(L^2(\mathbb{R}), L^2(\mathbb{R}))$.

(b) Find the operator norm $\|L\|$.

(c) Find the kernel and the range of L.

(d) Find the adjoint L^*.

(a) L is well-defined because if $f \in L^2(\mathbb{R})$, i.e. f is a measurable function defined on $\mathbb{R}$ so that $\int_{\mathbb{R}} |f(x)|^2 dx < \infty$, then Lf is again a measurable function on $\mathbb{R}$, and

$$\int_{\mathbb{R}} |(Lf)(x)|^2 dx = \int_0^\infty |f(x)|^2 dx + \int_{-\infty}^0 |f(-x)|^2 dx = 2\int_0^\infty |f(x)|^2 dx < \infty.$$

L is linear because for any $f, g \in L^2(\mathbb{R})$ and any scalars α and β,

$$L(\alpha f + \beta g)(x) = (\alpha f + \beta g)(|x|) = \alpha(Lf)(x) + \beta(Lg)(x).$$

L is continuous because it is bounded: for any $f \in L^2(\mathbb{R})$,

$$\|Lf\|_2 = \left(\int_{\mathbb{R}} |Lf(x)|^2 dx\right)^{1/2} = \left(2\int_0^\infty |f(x)|^2 dx\right)^{1/2} \le \sqrt{2}\|f\|_2.$$

(b) The last line above shows $\|L\| \le \sqrt{2}$. On the other hand, if we take f_0 to be a square integrable function on $\mathbb{R}$ that equals 0 for all $x < 0$, then

$$\|Lf_0\|_2 = \left(\int_{\mathbb{R}} |Lf_0(x)|^2 dx\right)^{1/2} = \left(2\int_0^\infty |f_0(x)|^2 dx\right)^{1/2} = \sqrt{2}\|f_0\|_2.$$

So $\|L\| \ge \sqrt{2}$. So $\|L\| = \sqrt{2}$.

(c) The kernel of L consists of all the measurable and square integrable functions on $\mathbb{R}$ which equals 0 for almost every $x > 0$.

The range of L consists of all the even functions on $\mathbb{R}$ that are measurable and square integrable.

(d) For any $f, g \in L^2(\mathbb{R})$,

$$(Lf, g) = \int_{\mathbb{R}} Lf(x)\overline{g(x)}dx = \int_0^{\infty} f(x)\overline{g(x)}dx + \int_{-\infty}^{0} f(-x)\overline{g(x)}dx$$
$$= \int_0^{\infty} f(x)\left(\overline{g(x) + g(-x)}\right) dx = (f, L^*g).$$

So the adjoint of L is

$$L^*g(x) = \begin{cases} g(x) + g(-x), & x \geq 0, \\ 0, & x < 0. \end{cases}$$

☐ 5. Prove the following statements:

(a) Let X be a topological vector space and let $L : X \to \mathbb{R}$ be a linear functional on X. Then L is continuous if and only if $ker(L)$ is closed.

(b) Any infinitely dimensional Frechét space contains a subspace that is not closed.

(a) If L is continuous, then $ker(L) = L^{-1}(0)$ is closed since $\{0\}$ is closed in $\mathbb{R}$.

Conversely suppose $ker(L)$ is closed, then $X/ker(L)$ is a topological vector space. More over, the map

$$\tilde{L} : X/ker(L) \to \mathbb{R}, \quad [x] = x + ker(L) \mapsto \tilde{L}([x]) = L(x)$$

is well-defined since

$$[x_1] = [x_2] \Longrightarrow x_1 - x_2 \in ker(L) \Longrightarrow L(x_1) = L(x_2) \Longrightarrow \tilde{L}([x_1]) = \tilde{L}([x_2]).$$

Obviously $\tilde{L}$ is linear and surjective. It is also injective since

$$\tilde{L}([x]) = 0 \Longrightarrow L(x) = 0 \Longrightarrow x \in ker(L) \Longrightarrow [x] = 0 \in X/ker(L).$$

So $\tilde{L}$ is a linear isomorphism from $X/ker(L)$ to $\mathbb{R}$. So $\tilde{L}$ is a homeomorphism, and in particular $\tilde{L}$ is continuous. So L is continuous since it is the combination of continuous maps $L = \tilde{L} \circ \pi$, where π is the canonical projection:

$$L : X \xrightarrow{\pi} X/\mathbb{R} \xrightarrow{\tilde{L}} \mathbb{R}.$$

(b) Let $\{x_1, x_2, x_3, \cdots\}$ be a countable sequence of linearly independent vectors in the given Frechét space. Let $X_n = span\{x_1, \cdots, x_n\}$ and let $X = \bigcup_{n=1}^{\infty} X_n$. Then X is a vector subspace of the given Frechét space. We claim theat X is not closed. Suppose X is closed, then it is complete, and thus a complete metric space. But each X_n is a finite dimensional vector subspace of X, which has to be closed.

Claim: Each X_n contains no interior points, since if X_n contains an interior point x_0, then 0 is also an interior point of $X_n = X_n - x_0$. In other words, X_n contains a neighborhood of 0 in X. As a consequence, X_n contains the whole of X since any neighborhood of 0 is absorbing. This conflicts with the fact that X_n is finite dimensional while X is infinite dimensional.

Back to the proof. We see that $X = \cup_n X_n$ is a countable union of nowhere dense subsets. So X is of first category. This conflicts with Baire's category theorem since (if X is closed, then) X is a complete metric space.

☐ 6. Prove the following statements:

(a) Let X, Y be Banach spaces, and $L : X \to Y$ is a continuous linear operator. Prove: If $x_n \xrightarrow{w} x$, then $Lx_n \xrightarrow{w} Lx$.

(b) In a Hilbert space H, a sequence x_n converges to x (in norm topology) if and only if $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$.

(a) Since L is a continuous linear operator defined on the whole of X, L^* is well-defined on the whole of Y^*. In other words, for any $y^* \in Y^*$, we have $L^* y^* \in X^*$. So

$$\langle Lx_n, y^* \rangle = \langle x_n, L^* y^* \rangle \xrightarrow{n\to\infty} \langle x, L^* y^* \rangle = \langle Lx, y^* \rangle.$$

So $Lx_n \xrightarrow{w} Lx$.

(b) If $x_n \to x$ in the norm topology, then $x_n \xrightarrow{w} x$ since the norm topology is stronger than the weak topology, and $\|x_n\| \to \|x\|$ since the norm function is continuous with respect to the norm topology.

Conversely, suppose $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$. Then $(x_n, x) \to (x, x) = \|x\|^2$ since the function $L_x : H \to \mathbb{F}, y \mapsto (y, x)$ is a linear functional, and $(x, x_n) \to \|x\|^2$ since $(x, x_n) = \overline{(x_n, x)}$. So we get

$$\|x_n - x\|^2 = (x_n - x, x_n - x) = \|x_n\|^2 - (x, x_n) - (x_n, x) + \|x\|^2 \xrightarrow{n\to\infty} 0.$$

So $x_n \to x$ in norm topology.

☐ 7. Let $X = l^\infty$ be the Banach space of all bounded sequences $x = (a_1, a_2, a_3, \cdots)$ of real numbers, with norm $\|x\| = \sup_n |a_n|$. Prove:

(a) There exists $L \in X^*$ with $\|L\| = 1$ such that if $x = (a_1, a_2, a_3, \cdots) \in X$ and $\lim a_n = a$ exists, then $L(x) = a$.

(b) There is no sequence $y = (b_1, b_2, b_3, \cdots) \in l^1$ such that $Lx = \sum_{n=1}^{\infty} a_n b_n$ for all $x \in X$.

(a) Let $Y \subset X$ be the subset

$$Y = \left\{ x = (a_1, a_2, a_3, \cdots) \in l^\infty \;\middle|\; \lim_{n\to\infty} a_n = a \text{ exists} \right\}.$$

Then Y is obviously a subspace of X. Define a linear functional l on Y by

$$l : Y \to \mathbb{F}, \quad l((a_1, a_2, \cdots)) = \lim_{n\to\infty} a_n.$$

The linearity of l is also trivial to check. It is continuous since

$$|l(x)| = |\lim_{n\to\infty} a_n| \le \sup_n |a_n| = \|x\|.$$

Note that if we choose $x = (a, a, \cdots)$ be the constant sequence, then $x \in Y$, $\|x\| = |a|$ and $|l(x)| = |a|$. So as a continuous linear functional on Y, we have $\|l\| = 1$. By the Hahn-Banach theroem, there exists $L \in X^*$ with $\|L\| = \|l\| = 1$ that extends l, i.e. if $x = (a_1, a_2, a_3, \cdots) \in l^\infty$ and $\lim a_n = a$ exists, then $L(x) = a$.

(b) Suppose such a $y = (b_1, b_2, \cdots) \in l^1$ exists. For each k we let $x_k = (0, \cdots, 0, 1, 0, \cdots)$, where 1 is at the kth entry and 0 is everywhere else. Then $x_k \in Y$ and $l(x_k) = 0$ for each k. So $Lx_k = 0$. But $Lx_k = \sum_n a_n b_n = b_k$. So $b_k = 0$ for each k. So $y = (0, 0, \cdots)$ and thus $Lx = 0$ for each x. This contradicts with the fact that $\|L\| = 1$.